

IL NUOVO CIMENTO

ORGANO DELLA SOCIETÀ ITALIANA DI FISICA

SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

VOL. I, N. 6

Serie decima

1° Giugno 1955

PRINTED IN ITALY

A Picture of the Electron.

M. BUNGE

Servicio Técnico Científico - Buenos Aires

(ricevuto il 28 Dicembre 1954)

Summary. — It is shown that, without performing a Foldy-Wouthuysen transformation, a position operator with a smooth motion can be defined for Dirac's electron. This mean-position operator may be interpreted as the center-of-mass operator. It is further shown that the vector product of the new mean-position operator by the kinetic momentum yields a tensor of mean-angular momentum, the 6 components of which are constants of the motion of the free electron. The equation of motion of the mean-angular momentum in the presence of an external field is shown to be the analogue of the classical torque equation. It is suggested that these results may be interpreted as showing that, while the electron's charge is concentrated at point \mathbf{x} , the electron's mass is spread over a region of dimensions of a Compton wave-length, this volume being the seat of well-known microcurrents associated with the intrinsic magnetic and electric moment. The advantages of this mixed picture of the electron over the point-particle model and over the extended-source model are finally discussed.

1. — A mean-position operator in the usual representation of Dirac's theory.

As is well-known, in the usual representation of Dirac's theory, the \mathbf{x} -coordinate performs a rapid oscillatory motion (*Zitterbewegung*). In the same representation there is a rather complicated operator the motion of which is not oscillatory, its time derivative being proportional to the momentum; this mean-position operator has been introduced by FOLDY and WOUTHUYSEN ⁽¹⁾,

⁽¹⁾ L. L. FOLDY and S. A. WOUTHUYSEN: *Phys. Rev.*, **78**, 29 (1950).

and reads

$$(1) \quad \mathbf{x}' = e^{-is} \mathbf{x} e^{is} = \mathbf{x} + \frac{i\hbar c}{2E_p} \boldsymbol{\gamma} + \frac{i\hbar c^2(\boldsymbol{\alpha}\mathbf{p})\boldsymbol{\alpha}}{2E_p(E_p + m_0c^2)}$$

where

$$(2) \quad e^{\pm is} = \frac{E_p + m_0c^2 \pm c\boldsymbol{\gamma}\mathbf{p}}{[2E_p(E_p + m_0c^2)]^{\frac{1}{2}}}, \quad E_p = c(p^2 + m_0^2c^2)^{\frac{1}{2}}, \quad \mathbf{p} = \frac{\hbar}{i} \nabla$$

and

$$(3) \quad \boldsymbol{\gamma} = \beta\boldsymbol{\alpha} = \begin{vmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{vmatrix}, \quad \beta = \gamma_0 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}.$$

But there is no need to perform a Foldy-Wouthuysen transformation in order to obtain a smoothly-moving co-ordinate. In fact, in the usual representation a much simpler operator with similar properties can be defined, namely

$$(4) \quad X_\mu^\lambda = x^\mu + \frac{\Lambda}{2} i\gamma^\mu, \quad \mu = 0, 1, 2, 3; \quad x^0 = ct, \quad \Lambda = \frac{\hbar}{m_0c}.$$

The space components of this linear operator are hermitean, and their eigenvalues are $x^i \pm \Lambda/2$. (Note that (4) is not a special case of (1), since Foldy-Wouthuysen's mean-position operator contains E_p , which is not a c -number but an integral operator.) In fact, recalling the derivation formula

$$(5) \quad \frac{dF}{dt} = \left[\frac{\partial}{\partial t} + \frac{i}{\hbar} H, F \right], \quad H = m_0c^2\beta + eA_0 + c\boldsymbol{\alpha}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right),$$

one obtains

$$(6) \quad m_0 \frac{dX_\mu}{dt} = \gamma_0 \Pi_\mu, \quad \Pi_\mu = p_\mu - \frac{e}{c} A_\mu,$$

Π_μ being the kinetic momentum.

That is to say, as a result of the cancelation of the separate tremblor movements of \mathbf{x} and of $(\Lambda/2)i\boldsymbol{\gamma}$, a position operator \mathbf{X} is obtained, the motion of which is smooth, since its velocity (6) is a diagonal operator ⁽²⁾. Notice that the motion of X_μ is smooth, but that dX_μ/dt is not a constant even in the absence of external fields, since $d\gamma_0/dt \neq 0$.

On the other hand, if we subject X^μ to a Foldy-Wouthuysen transform-

⁽²⁾ Here we are implicitly considering ordinary spinors, not containing Dirac matrices, so that for positive energy states $m_0 dX_\mu/dt$ equals the kinetic momentum, whereas for negative energy states it equals $-\Pi_\mu$. On the other hand, using spinors containing Dirac's matrices, as introduced by SAUTER, our interpretation — as well as FOLDY's conclusions — ceases to be valid, owing to the contribution of the fluctuating part of the spinors.

ation, we obtain an operator the motion of which is oscillatory. Indeed, the transforms of its components are

$$(7a) \quad X'_0 = e^{is} \left(ct + \frac{A}{2} i\gamma_0 \right) e^{-is} = x_0 + \frac{iA}{2E_p} (m_0 c^2 \beta - c\alpha p),$$

$$(7b) \quad \mathbf{X}' = e^{is} \left(\mathbf{x} + \frac{A}{2} i\boldsymbol{\gamma} \right) e^{-is} = \mathbf{x} + \frac{\hbar}{2m_0 E_p} [\boldsymbol{\sigma} \times \mathbf{p}] + \\ + i\hbar \frac{c^2(\alpha \mathbf{p})\alpha + (c/2m_0)(\boldsymbol{\gamma} \mathbf{p})\mathbf{p}}{2E_p(E_p + m_0 c^2)},$$

whose time derivatives are no longer diagonal.

2. — The corresponding mean-velocity operator.

Recalling that γ_0 is the operator representative of $(1 - v^2/c^2)^{\frac{1}{2}}$, it follows immediately from (6) that, in the absence of external fields, the average of $d\mathbf{X}/dt$ coincides with the macroscopic velocity:

$$(8) \quad \left\langle \frac{d\mathbf{X}}{dt} \right\rangle = \pm \mathbf{v}_{\text{macr}}.$$

This, added to eq. (6), suggests to call

$$(9) \quad V^\mu = \frac{dX^\mu}{dt} = \gamma_0 \Pi^\mu / m_0$$

the mean-velocity operator.

It seems easier to understand that v_{macr} is the mean value of a quantity whose instantaneous values are the eigenvalues of $\gamma_0 \Pi / m_0$ — that is, $\pm \Pi / m_0$ — than to accept the usual interpretation that v_{macr} is an average built out of instantaneous values $\pm c$. If, as is likely, the electron is to be regarded as having a complex structure, then it seems that V can be regarded as the operator representative of the velocity of the electron as a whole.

It is easy to see that the relative fluctuation (mean-standard deviation) of V increases with the velocity in accordance with

$$(10) \quad \frac{[(V - \bar{V})^2]^{\frac{1}{2}}}{\bar{V}} = \frac{v/c}{(1 - v^2/c^2)^{\frac{1}{2}}}, \quad \bar{V} \equiv \langle V \rangle, \quad v \equiv v_{\text{macr}}.$$

On the other hand, the relative fluctuation of $d\mathbf{x}/dt = c\alpha$ attains its maximum value (equal to 1) precisely for the electron at rest, a result which is at variance with the usual rule that, the less the velocity of a particle, the more it should

approach the classical behaviour. This seems to add strength to our suggestion that \mathbf{X} and $d\mathbf{X}/dt$ should be viewed as the correct quantal transcriptions of the classical position and velocity respectively.

As it is well-known, no satisfactory Dirac analogue of the Newton-Lorentz equation has as yet been proposed, since the acceleration

$$(11) \quad \frac{d^2 \mathbf{x}}{dt^2} = \frac{2c^2}{A} \left\{ i\boldsymbol{\gamma} - \left[\boldsymbol{\sigma} \times \frac{\boldsymbol{\Pi}}{m_0 c} \right] \right\},$$

is far from being proportional to the external force. On the other hand, the analogue of the Lorentz force, namely

$$(12) \quad \mathbf{F} = \frac{d\boldsymbol{\Pi}}{dt} = -e(\mathbf{E} + [\boldsymbol{\alpha} \times \mathbf{B}])$$

does not contain the acceleration. But our relation (6) enables us to obtain the following relation between the acceleration of the electron as a whole and the impressed Lorentz force:

$$(13) \quad m_0 \frac{d^2 \mathbf{X}}{dt^2} = \frac{d}{dt} (\gamma_0 \boldsymbol{\Pi}) = \gamma_0 \mathbf{F} - \frac{2c}{\hbar} (i\boldsymbol{\gamma} \boldsymbol{\Pi}) \boldsymbol{\Pi}.$$

The first term of the last member is clearly the analogue of the classical force, given by

$$(14) \quad \frac{d}{dt} \left[\frac{m_0 \mathbf{v}}{(1 - v^2/c^2)^{\frac{1}{2}}} \right] = \mathbf{F}.$$

The typical quantum term in (13) vanishes in the electron's rest frame (if no external field is present), and increases with the momentum; it may be described as a mechanical self-force producing the microaccelerations, or fluctuations in the velocity $d\mathbf{X}/dt$.

3. — New constants of the motion.

Let us introduce the displacement operator T defined by

$$(15) \quad T f(x^\mu) = f\left(x^\mu + \frac{A}{2} i\gamma^\mu\right) = f(X^\mu).$$

Obviously, T can be represented by

$$(16) \quad T = e^{iS}, \quad S = \frac{A}{2} \sum_{\mu} \gamma^\mu \frac{\partial}{\partial x^\mu},$$

which shows that it is unitary. Since T effects a shift from the trembling co-ordinate \mathbf{x} to the smooth co-ordinate \mathbf{X} , it may be expected that it will

«press» some otherwise «wrinkled» quantities, without having to change the representation of the theory — as in fact it does. Thus, by means of this operator we shall obtain 6 new constants of the motion, out of corresponding classical constants. Applying T to the cross product of x^μ into p_ν (which is a classical constant of the motion), we obtain ⁽³⁾ the 6 quantities

$$(17) \quad \mathbf{M} = T[\mathbf{x} \times \mathbf{p}] = \left[\left(\mathbf{x} + \frac{A}{2} i\boldsymbol{\gamma} \right) \times \mathbf{p} \right] = [\mathbf{X} \times \mathbf{p}],$$

$$(18) \quad \mathbf{N} = T(x_0 \mathbf{p} - \mathbf{x} p_0) = \left(x_0 + \frac{A}{2} i\gamma_0 \right) \mathbf{p} - \left(\mathbf{x} + \frac{A}{2} i\boldsymbol{\gamma} \right) p_0 = X_0 \mathbf{p} - \mathbf{X} p_0,$$

which are components of the skew-symmetric tensor

$$(19) \quad M_\nu^\mu = x^\mu p_\nu - x^\nu p_\mu + \frac{iA}{2} (\gamma^\mu p_\nu - \gamma^\nu p_\mu) = X^\mu p_\nu - X^\nu p_\mu.$$

We shall call this the tensor of the mean-angular momentum.

These quantities are constants of the motion in the absence of external fields, as is readily verified recalling the equation of motion (6) of the mean-position operator X^μ . Hence, instead of the 7 previously known constants of the motion with respect to time t , we now have 13 constants; these are the energy-momentum four-vector, the total angular momentum, and the tensor of the mean-angular momentum.

A somewhat tedious work shows that these 13 constants do not give rise to new ones; indeed, the 78 commutators that can be built out of them, either reproduce those constants or are simple combinations of them. Among these commutators, the following interesting four-pseudo-vector appears:

$$(20) \quad \delta_\mu = \gamma_0 \sigma_\mu + \sigma_0 \frac{p_\mu}{m_0 c}, \quad \mu = 0, 1, 2, 3,$$

where

$$(21) \quad \sigma_0 = \gamma_5 = \begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}.$$

The quantities (20) are constants of the motion in the field-free case, though not first constants. The cross product of $\boldsymbol{\delta}$ into \mathbf{p} yields what will be seen to be the magnetic partner of \mathbf{M} ; multiplied by the Bohr magneton, it reads

$$(22) \quad \boldsymbol{\eta} = [\boldsymbol{\mu} \times \mathbf{p}],$$

⁽³⁾ We are not performing a canonical transformation, since we do not wish to preserve the eigenvalues.

where

$$(23) \quad \boldsymbol{\mu} = \mu_0 \gamma_0 \boldsymbol{\sigma}, \quad \mu_0 = \frac{e\hbar}{2m_0c} = e \frac{A}{2},$$

is the operator of the intrinsic magnetic moment.

4. - Equation of motion of the mean-angular momentum in the presence of an external field.

The consideration of an external field will reinforce our interpretation of X^μ as a mean-position operator, and of M_ν^μ as the tensor of the mean-angular momentum. In the presence of an external field, (17) and (18) become

$$(24) \quad \mathbf{M} = [\mathbf{X} \times \boldsymbol{\Pi}],$$

$$(25) \quad \mathbf{N} = X_0 \boldsymbol{\Pi} - \mathbf{X} \Pi_0.$$

The equation of motion of these quantities are

$$(26) \quad \frac{d\mathbf{M}}{dt} = [\mathbf{X} \times \mathbf{F}],$$

$$(27) \quad \frac{d\mathbf{N}}{dt} = X_0 \mathbf{F}.$$

Clearly, these equations constitute the analogues of the classical theorem on the moment of the force. If \mathbf{F} is not the ponderomotive force, these equations need not contain the electric charge, so that they refer to the mechanical motion of the electron; moreover, they refer to the electron as a whole, since they contain the mean-position operator X^μ . In other words, eqs. (26) and (27) describe the action of external forces as if the electron were condensed in its center-of-mass \mathbf{X} . On the other hand, formula (12) for the Lorentz force shows that the electromagnetic field, which acts on the charge of the electron, is applied at the point \mathbf{x} . In short, while the charge is point-like, the mass seems to spread over a certain volume.

In contradistinction to classical mechanics, the torque does not vanish in the presence of central forces. Indeed, for $\mathbf{B} = 0$ and $\mathbf{E} \parallel \mathbf{x}$, a torque remains which equals the cross product of the intrinsic electric moment $\boldsymbol{\epsilon}$ into the electric field strength,

$$(28) \quad \frac{d}{dt} [\mathbf{X} \times \boldsymbol{\Pi}] = -\mu_0 [i\boldsymbol{\gamma} \times \mathbf{E}] = -[\boldsymbol{\epsilon} \times \mathbf{E}],$$

where

$$(29) \quad \boldsymbol{\epsilon} = \mu_0 i\boldsymbol{\gamma}.$$

Multiplying (26) by the electron charge, it can be read in terms of the total electric moment

$$(30) \quad eX = ex + \mu_0 i \gamma = ex + \epsilon.$$

The magnetic partner of eq. (26) is obtained from the value of (22) in the presence of an external field, namely

$$(31) \quad \eta = [\mu \times \Pi],$$

the time derivative of which is

$$(32) \quad \frac{d\eta}{dt} = [\mu \times F].$$

5. - A mixed picture of the electron.

The foregoing results can perhaps be used to build a new picture of the electron within the frame of the standard representation of the one-particle theory — a more or less visualizable picture which the physicist forms anyhow, despite the impossibility of visualizing matrices, and despite the restrictions imposed upon picturability by a certain philosophy. Essentially two pictures of the electron have been suggested in connection with Dirac's theory, namely the point-particle model and the extended-source model (¹); a third, mixed model, will now be proposed, in which the charge is point-like, while the mass is not.

The fact that in the usual representation the Lorentz force (eq. (12)) is applied at the terminus of the trembling vector x , suggests that the *charge* of Dirac's electron is located at that point, oscillating around the mean-position co-ordinate X with an amplitude of the order of λ . On the other hand, the following facts seem to suggest that the electron *mass* is spread over a region of dimensions of a Compton wavelength, a volume scanned by x and having no sharp boundaries: *a*) the mean-position operator, X , has a non-oscillatory motion (section 1); *b*) the average of dX/dt equals the macroscopic velocity (section 2); *c*) the mean-angular momentum $[X \times p]$ remains constant in the absence of external fields (section 3); and *d*) its rate of change in the presence of external fields equals the torque $[X \times F]$ (section 4).

When the electron moves as a whole, a redistribution of mass occurs and a self-force arises in accordance with the analogue (13) of the Newton-Lorentz equation. The region occupied by the electron is the seat of microcurrents the density of which is

$$(33) \quad j_{\text{micro}} = c \cdot \text{curl} (\psi^+ \mu \psi) + \frac{\partial}{\partial t} (\psi^+ \epsilon \psi),$$

as is well-known from the Gordon decomposition ⁽⁴⁾. Or, to put it into other words, the oscillatory motion of the charge, located at \mathbf{x} , gives rise to an intrinsic magnetic moment $\boldsymbol{\mu}$; in a system of reference in which the electron has an over-all motion, this magnetic moment engenders the electric moment $\boldsymbol{\epsilon}$.

6. — Discussion.

The most serious objection against the suggestion that the mass is spread over a volume is perhaps of a logical nature ⁽⁵⁾, namely, the fact that if Dirac's equation is derived as a linearization of Klein-Gordon's equation, the assumption is tacitly made that the electron is a mass-point — as seen from the hamiltonian form of the equation. But if Dirac's equation is taken as a fresh start, leaving aside its historical origin, then it seems permissible and advantageous to think of the electron's mass as occupying a small volume. This makes some properties of the electron picturable in semi-classical terms, and is probably more reasonable in view of the complex properties of the electron.

Simplicity and a certain degree of picturability — which after all are not criteria of truth — are not the sole advantages of the mixed model here proposed; two more advantages seem to speak in favour of it. In the first place, the statement — inferred from the formula $d\mathbf{x}/dt = c\boldsymbol{\alpha}$ — that the instantaneous velocity of the electron is $+c$ or $-c$, whatever its mean-velocity may be, is not only hard to understand but contradicts the relativistic statement that c is not attainable by particles (since such a velocity would require an infinite energy). If the mixed model is adopted, the instantaneous values of the velocity $d\mathbf{X}/dt$ of the electron as a whole are seen to be $\pm \mathbf{p}/m_0$ (section 2); since $c\boldsymbol{\alpha}$ is then the operator representative of the velocity of a mass-less point-charge, no contradiction with relativity is entailed in the assertion that the instantaneous velocity of the charge is $\pm c$.

In the second place, although $\mathbf{V} = d\mathbf{X}/dt$ is not a constant in free space, at least in the non-relativistic limit (where the mechanical self-force is small), it is nearly constant (cf. eqs. (10) and (13)); on the other hand, the fluctuation of $d\mathbf{x}/dt$ attains its peak precisely when the electron is macroscopically at rest. As a consequence, although neither the point-particle model nor the mixed model make room for the relativistic requirement of existence of inertial

⁽⁴⁾ Cf. L. DE BROGLIE: *Théorie Générale des Particules à Spin* (Paris, 1943), chap. V.

⁽⁵⁾ The fact that geometrical shape is not a Lorentz-invariant property, is not an argument against extended-source models, but against *rigid* models. In our picture no definite shape is assumed, so that considerations of Lorentz invariance would be out of place.

systems of reference, the mixed model seems a closer approximation to the satisfaction of this demand. At least it removes the paradox, arising in the point-particle model, that the electron moves more swiftly, the less its average velocity.

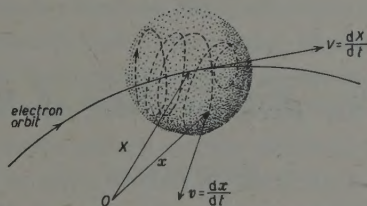


Fig. 1. — Classical picture of Dirac's electron. The curly motion of the charge is represented by the dashed line (x co-ordinate). The full line indicates the smooth motion of the center-of-mass (X co-ordinate). The order of magnitude of the electron diameter is one Compton wavelength.

Needless to say, the main disadvantage of the extended-source model is avoided in the mixed model, namely the need to postulate Poincaré stresses of an unknown nature in order to account for the stability of the electron, since in the mixed model the latter acts as an extended body in all but in the electromagnetic respects.

The classical picture suggested by the above remarks is shown in Fig. 1.

RIASSUNTO (*)

Si dimostra che senza eseguire una trasformazione di Foldy-Wouthuysen è possibile definire per l'elettrone di Dirac un operatore di posizione animato di moto non oscillatorio. Questo operatore di posizione media può essere interpretato come l'operatore del centro di massa. Si dimostra inoltre che il vettore prodotto del nuovo operatore di posizione media per il momento cinetico fornisce un tensore di momento angolare medio, le cui 6 componenti sono costanti del moto dell'elettrone libero. L'equazione del moto del momento angolare medio in presenza di un campo esterno si dimostra essere analoga all'espressione classica della coppia. Questi risultati si possono interpretare come favorevoli all'ipotesi che, mentre la carica dell'elettrone è concentrata nel punto x , la massa dell'elettrone è ripartita in una regione dalle dimensioni di una lunghezza d'onda Compton, il cui volume è sede di ben note microcorrenti associate coi momenti magnetico ed elettrico intrinseci. Si discutono finalmente i vantaggi di questa immagine mista dell'elettrone rispetto al modello della particella puntiforme e al modello della sorgente estesa.

(*) Traduzione a cura della Redazione.

Studies on the α -Particle Model of Nuclei. — I.

A. HERZENBERG

The Physical Laboratories - University of Manchester

(ricevuto il 14 Febbraio 1955)

Summary. — The kinetic energies of the $A=4n$ nuclei (with $Z=N=2n$) up to $A=40$ are estimated on the basis of the α -particle model. This estimate is used, together with known regularities in the binding energies and an estimate of the Coulomb energy, to show that, on this model, there must be an additive attractive interaction between neighbouring α -particles. New physical criteria are given for the stability of the α -particles and the degree to which their motion is adiabatic; the relevant parameters are estimated by the use of quantum mechanics. A comparison is made with the analogous situation in molecules.

1. — Introduction.

The possibility that nuclei might contain α -particles as sub-units has often been considered. Reviews of this work were given by ROSENFELD ⁽¹⁾ (p. 269), and BLATT and WEISSKOPF ⁽²⁾ (p. 292). This paper is intended as an introduction to a series dealing with various aspects of the α -particle model. The object here is, first, to add to an argument concerning the binding energies of some light nuclei, and, secondly, to estimate the physically important parameters of the α -particle model and to compare them with their analogues in molecules. Several of the ideas in this paper have been discussed before. The point of view adopted here is almost always new and the conclusions here do not always agree with those of the earlier discussions.

It was pointed out by WEFELMEIER ⁽³⁾ that if one pictured the light nuclei

⁽¹⁾ L. ROSENFELD: *Nuclear Forces* (Amsterdam, 1948).

⁽²⁾ J. M. BLATT and V. F. WEISSKOPF: *Theoretical Nuclear Physics* (New York, 1952).

⁽³⁾ W. WEFELMEIER: *Naturwiss.*, **25**, 525 (1937); *Zeits. f. Phys.* **107**, 332 (1937).

with $N = Z = 2n$ (n integral) and $A \leq 40$ as lattices of α -clusters (*), then the total binding energy in the ground state would be accounted for if, in addition to the internal binding energy of the α -clusters, one postulated a constant binding energy of about 2.4 MeV per neighbouring pair (bond). (This rule does not hold for the 2- α nucleus ${}^8\text{Be}$). HAFSTAD and TELLER (4) assumed an α - α interaction potential which increased quadratically with the displacements of the α 's from their relative equilibrium positions, and neglected the vibration-rotation interaction. They then found that in ${}^8\text{Be}$ ($n=2$), ${}^{12}\text{C}$ ($n=3$), and ${}^{16}\text{O}$ ($n=4$), the kinetic energy of the α -clusters was very nearly proportional to the number of bonds. Therefore, since the Coulomb interaction energy of the α -clusters in these cases is also proportional to the number of bonds, the α - α binding in ${}^{12}\text{C}$ and ${}^{16}\text{O}$ had, on this model, to be interpreted as being due to an additive nuclear interaction for each neighbouring α - α pair.

Estimates of the Coulomb energy for most of the $A = 4n$ nuclei ($N = Z$) have been made by BROWN and INGLIS (5). It turns out that the Coulomb energy is of the same order of magnitude as the binding energy of the α -clusters to one another.

In Section 2 it will be shown that the kinetic energy per vibrational degree of freedom in an infinite aggregate of α -clusters agrees well with estimates for ${}^{12}\text{C}$ and ${}^{16}\text{O}$, so that it is reasonable to assume this kinetic energy to be constant (**); it then becomes possible to compute the nuclear energy of interaction, which turns out to be very nearly proportional to the number of bonds (see table 2.1). Section 2 also contains a justification of the neglect of the energy contribution from resonance between different distributions of α -clusters over the lattice points. The α -particle model is therefore able to account for the binding energies of the $4n$ nuclei from $n=3$ to $n=10$ with a single free parameter, i.e. the expectation value of the nuclear interaction per bond. The calculation of the forces of interaction between α -particles will be taken up in succeeding papers.

Other evidence for a 4-periodicity of some sort comes from a plot of the binding energy of the last nucleon against A ; this plot shows pronounced peaks at $A = 4n$. The periodicity is still strong at $A = 40$ if one subtracts the Coulomb energy from the measured binding energies. The α -particle model would account for this periodicity by assuming the interaction between complete α -clusters and odd nucleons to be weak (as it clearly is in ${}^5\text{He}$). Other possi-

(*) We use the name « α -cluster» for an α -particle in a nucleus.

(4) L. HAFSTAD and E. TELLER: *Phys. Rev.*, **54**, 681 (1938).

(5) H. BROWN and D. R. INGLIS: *Phys. Rev.*, **55**, 1182 (1939).

(**) The number of vibrational degrees of freedom in 6 is ${}^{16}\text{O}$, and 24 in ${}^{40}\text{Ca}$. The kinetic energy per vibrational degree of freedom depends essentially only on the volume available to each α -cluster.

bilities which must be considered to account for the total binding energies and their variation with A are formulae of the v. Weizsäcker type and the symmetries of the wavefunction characteristic of some nuclear potentials.

A formula of the v. Weizsäcker type does not fit the binding energies in this region if one assumes the symmetry term to be a smooth function of A ⁽⁶⁾.

It can be shown ^(7,8) that if one assumes two-body central forces of predominantly Wigner and Majorana exchange types (that is forces depending only on the spatial co-ordinates of the nucleons), and the Coulomb forces are neglected, then one obtains the kinks in the binding energy curve without making any special assumption of spatial correlation. This approach meets with a difficulty due to the existence of tensor forces ⁽⁹⁾.

The semi-empirical « uniform » model ⁽⁸⁾ has recently been re-examined by BLATT and WEINBERG ⁽¹⁰⁾. (This model smooths out over all the details of the nuclear wavefunction except for those due to the symmetries which follow from a Hamiltonian depending only on the spatial co-ordinates of the nucleons). A difficulty is that if the model is adjusted to the isobaric energy differences in odd- A nuclei, then it probably underestimates the energy differences between even-even and odd-odd isobars. BLATT and WEISSKOPF ⁽¹¹⁾ interpret this result as implying some sort of close correlation between pairs of nucleons. With the best overall fit to the isobaric energy differences, the model predicts the right order of magnitude for the fluctuations in the binding energy of the last nucleon as A increases; but if one goes further and assumes the nuclear forces to satisfy the saturation conditions, then one obtains too small an upper limit for the total binding energy ⁽¹⁰⁻¹²⁾.

FEENBERG and WIGNER ⁽¹³⁾ and FEENBERG and PHILLIPS ⁽¹⁴⁾ have calculated the binding energies of the nuclei from ${}^5\text{He}$ to ${}^{16}\text{O}$ with independent particle wavefunctions. They assumed central forces of predominantly Wigner and Majorana exchange type. Similar calculations for the same range of nuclei have been made by KURATH ⁽¹⁵⁾ with independent particle wavefunctions and a strong interaction coupling the spin and orbital angular momenta of each particle; Wigner and Majorana central forces were treated as a perturbation.

⁽⁶⁾ N. FEATHER: *Advances in Physics*, **2**, 141 (1953).

⁽⁷⁾ E. P. WIGNER: *Phys. Rev.*, **51**, 106 (1937).

⁽⁸⁾ E. P. WIGNER: *Phys. Rev.*, **51**, 947 (1937).

⁽⁹⁾ E. FEENBERG and E. P. WIGNER: *Rep. Prog. Phys.*, **8**, 274 (1941).

⁽¹⁰⁾ J. M. BLATT and I. G. WEINBERG: *Am. Journ. of Phys.*, **21**, 124 (1953).

⁽¹¹⁾ J. M. BLATT and V. F. WEISSKOPF: *Theoretical Nuclear Physics* (New York, 1952).

⁽¹²⁾ D. FRISCH: *Phys. Rev.*, **84**, 1169 (1951).

⁽¹³⁾ E. FEENBERG and E. P. WIGNER: *Phys. Rev.*, **51**, 95 (1937).

⁽¹⁴⁾ E. FEENBERG and M. PHILLIPS: *Phys. Rev.*, **51**, 597 (1937).

⁽¹⁵⁾ D. KURATH: *Phys. Rev.*, **88**, 804 (1952).

The results of both calculations showed peaks in the binding energy of the last nucleon at $A=8, 12$, and 16 . These peaks were less pronounced in the presence of the spin-orbit interaction. In both cases the total binding energy turned out to be much too small.

The physical basis of the α -particle model has been discussed by WHEELER⁽¹⁶⁾. According to WHEELER, the condition that the α -particle model be valid is that the frequency $(\omega_0/2\pi)$ — the frequency of vibration or rotation of an aggregate of α -clusters, — should be very much larger than $(\omega_x/2\pi)$ — the frequency with which an α -cluster exchanges nucleons with its neighbours. WHEELER⁽¹⁶⁾ also suggested that the α - α interaction must be essentially due to the exchange of nucleons. Section 4 contains a discussion of the physical basis which differs from WHEELER's on two points. First, WHEELER's existence condition seems to be too restrictive; an alternative condition suggested here is $\omega_{\text{int}} \gg \frac{1}{4}\omega_x$ where ω_{int} is the internal vibration frequency of a nucleon in an α -particle. (This condition implies that a nucleon must carry out several complete oscillations in an α -cluster before passing out through the surrounding potential barrier). Secondly, we use wave mechanics to estimate ω_x . (WHEELER pointed out that his classical discussion was not justified). It will also be shown that the assumption of nucleon exchange forces between α -clusters is inconsistent with the assumption that an α -cluster should be stable for times long compared with the periods of collective nuclear motion. The estimate given here of ω_x brings out the importance of the radius of the α -particle in the theory.

Section 4 contains also a comparison of the nuclear α -particle model with analogous problems in molecular theory. It turns out that the nucleon exchange between α -clusters should be relatively much less important than electron exchange in molecules; that the adiabatic approximation is much better justified in molecules than in the α -particle model; that rotation-vibration interaction in α -nuclei must be important; and that the neglect of the anharmonic character of the α - α forces is not justified.

A discussion by DENNISON⁽¹⁷⁾ of the energy level spectrum of the ^{16}O nucleus on the basis of the α -particle model appeared after most of this manuscript had been completed. The agreement between theory and experiment is remarkable. Nevertheless, the result must be treated with caution because, as Dennison points out, the neglect of the rotation-vibration interaction and the anharmonicity is not justified without further discussion. Moreover, although DENNISON used bound-state theory, a number of his levels is unstable with respect to α -particle emission. An analysis of the ^8Be , ^{12}C , and ^{16}O energy level spectra with more limited data had previously been made by INGLIS⁽¹⁸⁾,

⁽¹⁶⁾ J. A. WHEELER: *Phys. Rev.*, **52**, 1083 (1937).

⁽¹⁷⁾ D. M. DENNISON: *Phys. Rev.*, **96**, 378 (1954).

⁽¹⁸⁾ D. R. INGLIS: *Rev. Mod. Phys.*, **25**, 390 (1953).

also with neglect of anharmonicity, rotation-vibration interaction and lastly of the virtual character of the levels. A first attack on the problem of these approximations has been made by HAEFNER ^(18a) who solved the Schrödinger equation for two α -particles and showed that an interaction potential of reasonable shape could account for the existence of a virtual 2^+ level at an excitation energy of about 3 MeV.

LOPES and GOLDEMBERG ⁽¹⁹⁾ have analysed the energy variation of the photon absorption cross-section of a number of nuclei and claim to have found evidence for the α -particle model. Their work is based on a formula due to LEVINGER and BETHE ⁽²⁰⁾ for the harmonic mean energy of absorbed photons. This formula is correct for an α -particle model of the nucleus if the (proton mass centre) — (neutron mass centre) vector-separations in different α -clusters are not correlated. However, LEVINGER and KENT ⁽²¹⁾ have shown that if one takes the Pauli principle correlations in an independent particle model into account (this was not done by LEVINGER and BETHE), then the formula used by LOPES and GOLDEMBERG has to be modified. If in addition to this modification the radius of a nucleus of mass number A is taken to be $1.2A^{\frac{1}{3}} \cdot 10^{-13}$ cm instead of $1.5A^{\frac{1}{3}} \cdot 10^{-13}$ cm as assumed by LOPES and GOLDEMBERG, the inconsistency with an independent particle model is considerably reduced.

A serious difficulty of the α -particle model is its apparent inconsistency with the independent particle shell model (see e.g. PRYCE ⁽²²⁾). However, the evidence for the shell model concerns only the last shell to fill. There is no direct evidence that the part of the wavefunction describing the nucleons in the inner shells retains its independent particle character.

2. — The Binding Energies of the $4n$ Nuclei.

WEFELMEIER ⁽³⁾ pointed out that if one thought of these nuclei as lattices of α -clusters (see however footnote on p. 995), then the binding energies could be interpreted as being due to the internal binding of the α -clusters, plus an α - α binding of about 2.4 MeV per bond. When we come to consider these regularities in more detail, it will be necessary to estimate the kinetic energy of the α 's and the energy of exchange resonance due to the different possible distributions of the α -clusters over the lattice points. To make these esti-

^(18a) R. R. HAEFNER: *Rev. Mod. Phys.*, **23**, 228 (1951).

⁽¹⁹⁾ J. GOLDEMBERG and L. LEITE LOPES: *Nuovo Cimento*, **12**, 817 (1954).

⁽²⁰⁾ J. S. LEVINGER and H. A. BETHE: *Phys. Rev.*, **85**, 577 (1952).

⁽²¹⁾ J. S. LEVINGER and D. C. KENT: *Phys. Rev.*, **95**, 418 (1954).

⁽²²⁾ M. H. L. PRYCE: *Rep. Prog. in Phys.*, **17**, 1 (1954).

mates, we need the average separation s_α of two neighbouring α -clusters; we now proceed to estimate this quantity.

In a large $(4n)$ nucleus of radius $r_0(4n)^{\frac{1}{3}}$, the volume available to each α -cluster is $(16\pi r_0^3/3)$. Therefore we have for a cubic lattice

$$(2.1) \quad s_\alpha \cong (16\pi r_0^3/3)^{\frac{1}{3}} = 2.56r_0 = 3.71 \cdot 10^{-13} \text{ cm}$$

and for the lattice corresponding to the closest packing of spheres

$$(2.2) \quad s_\alpha \cong 2^{\frac{1}{3}} \cdot 3.71 \cdot 10^{-13} \text{ cm} = 4.17 \cdot 10^{-13} \text{ cm}.$$

We take $r_0 = 1.45 \cdot 10^{-13} \text{ cm}$.

For the $4n$ nuclei with $n = 2$ (^8Be), $n = 3$ (^{12}C) and $n = 4$ (^{16}O), s_α can be estimated from the position of excited levels if we assume that the excitation energy is entirely kinetic energy of rotation. The ^8Be nucleus has a number of α -unstable states in the region from 2 to 3 MeV, at least one of which is 2^+ ($^{23, 24}$). According to the α -particle model, the ^8Be nucleus is a dumbbell with an 0^+ ground state and 2^+ , 4^+ , ... rotational excited states. If we take the first excited state to be 2.9 MeV (2^+), we obtain

$$(2.3) \quad s_\alpha (^8\text{Be}) \cong 4.65 \cdot 10^{-13} \text{ cm}.$$

This result may be an underestimate.

The ^{12}C nucleus has a 2^+ excited state at 4.44 MeV (23) in which the triangle of α -cluster rotates about an axis passing through one vertex and the centre of the opposite side. We have

$$(2.4) \quad s_\alpha (^{12}\text{C}) \cong 3.76 \cdot 10^{-13} \text{ cm}.$$

The ^{16}O nucleus is to be pictured as a tetrahedron with α -clusters at its vertices. There is a 3^- level of 6.13 MeV excitation energy in which the tetrahedron rotates about an axis perpendicular to one of the four faces and passing through the opposite vertex ($^{17, 18, 25}$). We obtain

$$(2.5) \quad s_\alpha (^{16}\text{O}) \cong 3.2 \cdot 10^{-13} \text{ cm}.$$

In the rest of this paper we take the value of s_α to be $3.7 \cdot 10^{-13}$.

The total energy of a $(4n)$ nucleus in the ground state is

$$(2.6) \quad E_0^{(n)} = nE_0^{(1)} + \langle K \rangle + \langle V \rangle + \langle C \rangle,$$

(23) F. AJZENBERG and T. LAURITSEN: *Rev. Mod. Phys.*, **24**, 321 (1952).

(24) N. GLÄTTLI, E. LOEPFE and H. WIDMER: *Helv. Phys. Acta*, **27**, 182 (1954).

(25) D. M. DENNISON: *Phys. Rev.*, **57**, 454 (1940).

where $(-E_0^{(1)})$ is the binding energy of an α -particle, and $\langle K \rangle$, $\langle V \rangle$, $\langle C \rangle$ respectively are the expectation values of the relative kinetic energy of the clusters, and their nuclear and coulomb energies of interaction. For the moment we neglect the exchange resonance.

Further on we estimate $\langle K \rangle$ and $\langle C \rangle$ to obtain an estimate of $\langle V \rangle$ from the experimental measurements of $E_0^{(n)}$, $E_0^{(1)}$. The results are given in Table 2.1. The nucleus ${}^8\text{Be}$ is anomalous and is not taken into account in the considerations which follow. Columns (1) and (2) give the name and number of α 's of the nuclei. (We have added the nuclei ${}^{36}\text{A}$ and ${}^{40}\text{Ca}$ to the list given by WEFELMEIER). Column (3) gives b , the number of bonds. Column (4) is explained below. Column (5) gives an estimate of $\langle K \rangle$, and columns (6) and (7) of $\langle C \rangle$. Column (8) contains the experimental values of the binding energies of α -particles to one another. (The values have been taken from LI *et al.* ^(26, 27) for A up to 32; for $A=36$ and $A=40$ we have used the values given by ENDT and KLUYVER ⁽²⁸⁾). Column (9) gives $\langle V \rangle$. Column (10) gives the deviation of the values in column (9) from the best straight line (least squares) through the origin on a plot of $\langle V \rangle$ against b . We see that $\langle V \rangle$ is represented fairly accurately by the formula $\langle V \rangle_{\text{calc}} = 5.2063 b \text{ MeV}$, which implies an additive interaction between the adjacent α -clusters. The sum of the squares of the deviations in column (10) is given at the bottom of that column. There is no clear systematic variation of $(\langle V \rangle - \langle V \rangle_{\text{calc}})$ with b .

To get some idea of the significance of the regularity in $\langle V \rangle$, a least squares straight line fit has been made on a plot of $\langle V \rangle$ against A . The resulting best straight line is $\langle V \rangle = (4.5158 A - 42.907) \text{ MeV}$. The deviations of $\langle V \rangle$ from this straight line are given in column (11), and the sum of the squares of these deviations is at the bottom of that column. The deviations are clearly systematic, and the sum of the squares of the deviations is nearly three times as large as for the formula $\langle V \rangle \propto b$. ($\langle K \rangle$ is linear in A . Therefore its inclusion does not affect the linearity in the plot of $\langle V \rangle$ against A .)

The errors quoted in column (8) are the errors due to the mass defect in the nuclei to which the different lines belong; a systematic error proportional to A is due to the probable error of $\pm 0.014 \text{ MeV}$ in the mass defect of the α -particle and has not been included.

The estimate of $\langle C \rangle$ is made on the basis of an assumed nuclear radius of $1.45 A^{1/3} \cdot 10^{-13} \text{ cm}$. If the nuclear radius is taken to be $1.2 A^{1/3} \cdot 10^{-13} \text{ cm}$, the best formula for $\langle V \rangle \propto b$ becomes $\langle V \rangle_{\text{calc}} = 5.6000 b \text{ MeV}$; the sum of the squares of the deviations of $\langle V \rangle$ from this formula increases from 18.2 to 24.7 MeV^2 .

⁽²⁶⁾ C. W. LI: *Phys. Rev.*, **88**, 1038 (1952).

⁽²⁷⁾ C. W. LI, W. WHALING, W. A. FOWLER and C. C. LAURITSEN: *Phys. Rev.*, **83**, 512 (1951).

⁽²⁸⁾ P. M. ENDT and J. C. KLUYVER: *Rev. Mod. Phys.*, **26**, 95 (1954).

TABLE 2.1.

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) |
|------------------|-----|-----|-----|---------------------|---|---|----------------------------------|----------------------------|---|---|
| Nucleus | n | b | f | $\langle K \rangle$ | $\langle O \rangle$ equ. (28) MeV | $\langle O \rangle$ (simple lattice) MeV | $-E_0^{(n)} + nE_0^{(1)}$ MeV | $\langle V \rangle$ MeV | $\langle V \rangle$ — 5.2063b MeV | $\langle V \rangle$ — [4.5158A — 42.907] MeV |
| ^8Be | 2 | 1 | 1 | 1.1 | 1.762 | 1.55 | — 0.096 \pm .027 | 2.766 | — | — |
| ^{12}C | 3 | 3 | 3 | 3.3 | 4.860 | 4.66 | 7.278 \pm .015 | 15.438 | — 0.181 | + 4.155 |
| ^{16}O | 4 | 6 | 6 | 6.6 | 9.119 | 9.33 | 14.426 \pm .000 | 30.145 | — 1.093 | + 0.799 |
| ^{20}Ne | 5 | 9 | 9 | 9.9 | 14.430 | 14.93 | 19.172 \pm .019 | 43.502 | — 3.355 | — 3.907 |
| ^{24}Mg | 6 | 12 | 12 | 13.2 | 20.717 | — | 28.504 \pm .024 | 62.421 | — 0.055 | — 3.051 |
| ^{28}Si | 7 | 16 | 15 | 16.5 | 27.923 | — | 38.499 \pm .030 | 82.922 | — 0.379 | — 0.613 |
| ^{32}S | 8 | 19 | 18 | 19.8 | 36.000 | — | 45.443 \pm .039 | 101.243 | + 2.323 | — 0.356 |
| ^{36}Ar | 9 | 23 | 21 | 23.1 | 44.910 | — | 52.059 \pm 0.1 | 120.069 | + 0.324 | + 0.407 |
| ^{40}Ca | 10 | 27 | 24 | 26.4 | 54.624 | — | 59.266 \pm 0.1 | 140.290 | — 0.280 | + 2.565 |
| | | | | | | | | | $\Sigma \Delta^2 = 18.15$ MeV | $\Sigma \Delta^2 = 49.72$ MeV ² |

Such a reduction of the nuclear radius would increase the curvature on a plot of $\langle V \rangle$ against A and make the straight line fit there worse.

We now proceed to derive the estimates of $\langle K \rangle$ and $\langle C \rangle$ used in Table 2.1.

The ground states of the $4n$ nuclei all have zero angular momentum so that $\langle K \rangle$ is entirely vibrational. We denote the number of vibrational degrees of freedom by f , and therefore have $f=1$ for $n=2$, $f=3n-6$ for $n \geq 3$. The values of f are given in column (4) of Table 2.1.

In the instance of ^{16}O , the energy level structure of the tetrahedron of α -clusters has been investigated by DENNISON⁽²⁵⁾ on the assumption that the clusters carry out vibrations of small amplitude about their equilibrium positions to which they are bound by parabolic potential forces. INGLIS⁽¹⁸⁾ has fitted the three vibrational frequencies and the separations to the first four known levels, and from his results we find (in DENNISON's notation) that $\hbar\omega_1 = 6.05$ MeV, $\hbar\omega_2 = 3.85$ MeV, $\hbar\omega_3 = 5.08$ MeV. The mean zero point kinetic energy per vibrational degree of freedom is $(\langle K \rangle/f) = 1.21$ MeV. (The figure $\hbar\omega_3 = 4.24$ MeV quoted on p. 434 of INGLIS' paper should be 5.08. « Identification (b) » of DENNISON's paper⁽¹⁷⁾ is consistent with additive α - α interactions and leads to 1.17 MeV per vibrational degree of freedom.)

Using the force constants for the stretching of nearest neighbour bonds obtained from this analysis, INGLIS⁽¹⁸⁾ (see also⁽⁴⁾), estimates the zero point kinetic energies for ^8Be and ^{12}C to be 1.07 MeV and 3.17 MeV respectively. Thus the zero point kinetic energy per vibrational degree of freedom in ^8Be is 1.07 MeV, and in ^{12}C it is 1.06 MeV.

In the heavier ($4n$) nuclei with $5 \leq n \leq 10$, $(\langle K \rangle/f)$ can be expected to lie between the values obtained above and the limit attained for $n \rightarrow \infty$. This latter may be estimated from an α -cluster in a parabolic potential such that its R.M.S. displacement $(\overline{x^2})^{1/2}$ from the origin along each of three mutually perpendicular axes is equal to that of a uniform distribution over a cube of side $s_\alpha = 3.71 \cdot 10^{-13}$ cm with the origin at the centre (see (2.1)). We obtain

$$(2.7) \quad \lim_{n \rightarrow \infty} (\langle K \rangle/f) = \frac{\hbar^2}{2.4m} \cdot \frac{1}{4s_\alpha^2} = \frac{3}{8} \cdot \frac{\hbar^2}{ms_\alpha^2} = 1.13 \text{ MeV},$$

where m is the nucleon mass. Since (2.7) agrees well with the estimates for ^8Be , ^{12}C , ^{16}O , we take the zero point kinetic energy per vibrational degree of freedom to be 1.1 MeV for all ($4n$) nuclei in column (5) of Table 2.1.

The Coulomb interaction energy $\langle C \rangle$ for $n > 2$ is calculated as follows. If the charge of each α -cluster were distributed uniformly over the volume it occupies, a ($4n$) nucleus would become a uniformly charged sphere of radius $r_0(4n)^{1/3}$. The total Coulomb energy of this is $(3/5)[(2ne)^2/r_0(4n)^{1/3}] = 1.50n^{2/3}$. This Coulomb energy may be divided into the self energy of the charge distribution within each α -volume, and the coulomb interaction energy of these

separate volumes. The latter is approximately equal to $\langle C \rangle$, and for the former we use n times the Coulomb energy of a sphere whose volume is equal to that occupied by an α -cluster,

$$n \cdot \frac{3}{5} \cdot \frac{(2e)^2}{4^{\frac{1}{3}} r_0} = 1.50 n \text{ MeV}.$$

In this way we obtain

$$(2.8) \quad \langle C \rangle = 1.50(n^{\frac{1}{3}} - n) \text{ MeV}.$$

For $n=3, 4, 5$ we also calculate $\langle C \rangle$ for Wefelmeier's structures, assuming $s_\alpha = 3.7 \cdot 10^{-13}$ cm. The results differ only slightly from those obtained by the approximation (2.8) which should improve in accuracy as n increases (*).

In counting bonds we have only taken nearest neighbours into consideration. However, as WEFELMEIER (3) shows for his structures, the distances between non-neighbours are appreciably larger than those between neighbours; e.g. in the triagonal bipyramid of ^{20}Ne , the distance between the pyramid

(*) The assumption of an α -lattice is not essential to the foregoing argument, and for the bigger ($4n$) nuclei a « liquid » of α -clusters would be consistent with the evidence. Of the three quantities $\langle C \rangle$, $\langle K \rangle$, b which enter into the discussion, $\langle C \rangle$, being the expectation value of a long-ranged potential, is not sensitive to the details of the wave function, and the estimate of $\langle K \rangle$ required only the volume available to each α -cluster and the fact, which follows from the exclusion principle, that the α -clusters do not overlap. Therefore any model giving bond numbers b proportional to those for the lattice will give a constant nuclear energy per bond. For example, one can calculate b_L for a model in which the α -clusters move without correlation within a spherical volume of radius $1.45(4n)^{\frac{1}{3}} \cdot 10^{-13}$ cm, each α - α pair with a separation smaller than some typical value s_0 being counted as a bond. Using the formulae of BLATT and WEISSKOPF (11)(p. 123), we find the following results for the ratio (b_L/b_W) , with $s_0 = 5.0 \cdot 10^{-13}$ cm:

| $n =$ | 3 | 4 | 5 | 6 | 7 | 9 | 8 | 10 |
|-------------|------|------|------|------|------|------|------|------|
| $b_L/b_W =$ | 0.89 | 0.79 | 0.79 | 0.81 | 0.79 | 0.81 | 0.80 | 0.80 |

(In a lattice the surface of a sphere of radius $s_0 = 5.0 \cdot 10^{-13}$ cm with a given α -cluster as centre would fall approximately half way between the nearest and next nearest neighbours.)

Our estimates for s_α in lattices are not very sensitive to the type of lattice [(2.1), (2.2)]; the estimates are unlikely to be seriously in error if the lattice assumption is dropped altogether.

Whether the α -clusters form a lattice or a less ordered structure depends on the details of the α - α forces, and the exchange resonance frequencies (see text) to which they give rise. The effect of exchange resonance is to loosen up the lattice structure.

apices is $1.63 s_\alpha$, and in the tetragonal bipyramid of ^{24}Mg it is $1.41 s_\alpha$. Hence the expectation value of an additive potential for these pairs will be smaller than the contribution from a bond by a factor of about $\exp[-0.5 s_\alpha/\lambda]$ (where λ is the range of the interaction potential). For $s_\alpha \cong 3.7 \cdot 10^{-13}$ cm and $\lambda \cong 10^{-13}$ cm we have $\exp[-0.5 s_\alpha/\lambda] \cong 0.16$.

So far we have assumed that the α -clusters lie on a well-defined lattice. This is impossible for $n > 4$, for the α -clusters obey Bose statistics and not all the lattice positions are then equivalent. For instance ⁽²⁹⁾ the nucleus ^{20}Ne (triangular bipyramid) would resonate between a configuration where α -clusters 1, 2 are at the pyramidal apices and α -clusters 3, 4, 5 in the central plane, and others in which, for example, α -clusters 1 and 3 have changed places.

This process gives rise to an additional term in the energy which we shall now show to be small. We obtain a rough estimate of this extra term from the exchange energy of an α -cluster resonating between two potential troughs in the regions $x_1 < x < x_2$, $x_3 < x < x_4$ separated by a potential barrier in $x_2 < x < x_3$. It can be shown by use of the WKB approximation that the energies E of the states which have even and odd parity about the mid point of the barrier are respectively given by

$$(2.10) \quad \int_{x_1}^{x_2} k dx \simeq (s + \tfrac{1}{2})\pi \mp \tfrac{1}{2} \exp[-G],$$

where

$$(2.11) \quad k(x) = \left\{ \frac{2 \cdot 4m}{\hbar^2} [E - V(x)] \right\}^{\frac{1}{2}}$$

and

$$(2.12) \quad G = \int_{x_3}^{x_4} \left\{ \frac{8m}{\hbar^2} [V(x) - E] \right\}^{\frac{1}{2}} dx.$$

s is a positive integer or zero, and x_1, x_2, x_3, x_4 are the turning points. The integral (2.12) extends through the region where $V > E$. When (2.10) is applied to a pair of wells in which the potential is either parabolic or increasing linearly with the modulus of the displacement from the respective centre, (2.10) takes the form

$$(2.13) \quad E \cong E_s \mp \frac{D}{2\pi} \exp[-G].$$

⁽²⁹⁾ E. TELLER and J. A. WHEELER: *Phys. Rev.*, **53**, 779 (1938).

Here E_s is the s -th energy level in a single well of the particular shape and $D \cong (E_{s+1} - E_s)$, i.e. the level spacing. We assume that (2.13) is valid for all reasonable well shapes. Thus we have for the exchange energy

$$(2.14) \quad \langle E_x \rangle \cong \mp \frac{D}{2\pi} e^{-q}.$$

TELLER and WHEELER (²⁹) have shown that in a lattice of α -clusters, (2.14) has still to be multiplied by a factor due essentially to the number of exchange resonances present. This factor also depends on the rotational quantum numbers of the state of the system; it is 3 for the ground state of ^{20}Ne ($\nu=5$) if only the easiest exchanges (lowest barrier) are taken into account.

For an estimate of (2.14), we put $D = 4 \cdot 1.1 = 4.4$ MeV, $(V - E) \cong 5$ MeV in the barrier, and take the barrier length to be $s_\alpha = 3.7 \cdot 10^{-13}$ cm. We then obtain

$$|\langle E_x \rangle| \cong \frac{4.4}{2\pi} e^{-3.65} = 0.018 \text{ MeV}.$$

We now follow TELLER and WHEELER (²⁹) and calculate (2.14) for a potential which increases parabolically from the centres of the two troughs, the two parabolae being joined at the centre of the barrier to form a sharp peak. The parameter of the parabolae is determined by requiring that an α -cluster vibrating in a single such trough should have a frequency ω . If we assume the barrier to be sufficiently large, and put $E = \frac{1}{2}\hbar\omega$ in (2.12), we find for the exchange energy

$$|\langle E_x \rangle| \cong \frac{3}{2\sqrt{\pi}} \hbar\omega \left(\frac{4m\omega}{\pi\hbar} \right)^{\frac{1}{2}} \left(\frac{l}{2} \right) \exp \left[- \left(\frac{l}{2} \right)^2 \left(\frac{4m\omega}{\hbar^2} \right) \right],$$

where l is the barrier length. This formula is identical with that given by TELLER and WHEELER, which was calculated in a different way, except for the factor $(3/2\sqrt{\pi}) = 0.846$. If we insert $l \cong s_\alpha = 3.7 \cdot 10^{-13}$ cm, and put $\hbar\omega = 4.4$ MeV, we obtain

$$|\langle E_x \rangle| = 0.05 \text{ MeV}.$$

TELLER and WHEELER took the barrier length l to be the square root of the sum of the squares of the displacements of the α -particles during the displacement of the lattice from one to the other equivalent position. For ^{20}Ne , they thus find $l = (0.74)^{\frac{1}{2}} s_\alpha$. By putting $\hbar\omega = 1.12$ MeV (which seems too small), and $s_\alpha = 9.1 \cdot 10^{-13}$ cm (too large) they obtain $|\langle E_x \rangle| \cong 0.16$ MeV.

Taking into account the factor due to the number of exchange resonances

we see that the exchange energy in the ^{20}Ne lattice is probably in the range 0.05 to 0.2 MeV. In the other α -nuclei the exchange energy must be expected to be of the same order of magnitude.

3. - The Size of the α -Particle.

The value of the radius of the α -particle will occur in the argument at several points. To find this quantity, we use the following approximation for the density distribution in an α -particle:

$$(3.1) \quad \rho(r) = C, \quad 0 \leq r \leq R_0$$

$$(3.2) \quad = C \left(\frac{R_0}{r} \right)^2 \exp \left[- \frac{2(r - R_0)}{\mu} \right], \quad r \geq R_0.$$

Here r is the distance from the mass centre, C is a normalisation constant, and R_0 , μ are constants to be determined.

It is easily shown that the form (3.2) is given by the asymptotic behaviour of the α -particle wave function when a nucleon is separated from the other three. The constant μ is

$$(3.3) \quad \mu = \frac{3}{4} \left(\frac{3m |E_0^{(1)} - E^{(t)}|}{2\hbar^2} \right)^{-\frac{1}{2}} = 0.87 \cdot 10^{-13} \text{ cm},$$

where $E_0^{(1)}$ is the ground state energy of the α -particle, and $E^{(t)}$ that of the triton.

We shall adjust the constant R_0 in (3.2) so that (3.1) and (3.2) may lead to the correct RMS value $(r^2)^{\frac{1}{2}}$ of the variable r . We define the radius r_α of the α -particle as the radius of a sphere of uniform density having the correct R.M.S. radius. (We make this arbitrary choice for the identification of r_α and R_0 because $(\bar{r}^2)^{\frac{1}{2}}$ is well-defined, and usually fairly easily calculable for all wave functions.) We have $r_\alpha = 1.29(r^2)^{\frac{1}{2}}$. The relation between R_0 and $(\bar{r}^2)^{\frac{1}{2}}$ is shown in Fig. 3.1.

An indication of the value of r_α can be obtained by comparing the results of calculations on the scattering of nucleons by α -particles with the experimental measurements. HEIDMANN⁽³⁰⁾ has calculated the scattering of 90 MeV neutrons by α -particles in Born approximation. His results are probably most accurate for the elastic scattering, which is strongly peaked in the forward direction, and we therefore confine our considerations to this. HEIDMANN

(30) J. HEIDMANN: *Phil Mag.*, **41**, 444 (1950).

assumed a Gaussian wavefunction for the α -particle, with $r_\alpha = 2.13 \cdot 10^{-13}$ cm, and a nucleon-nucleon potential of the Serber form

$$(3.3) \quad V_{ij} = -\frac{1}{2} \mathcal{G} (1 + P_{ij}^x) (1 - g + g P_{ij}^\sigma) \exp[-\kappa r_{ij}^2],$$

where P_{ij}^x , P_{ij}^σ are the operators exchanging spatial and spin co-ordinates respectively, and $\kappa^{-\frac{1}{2}} = 1.94 \cdot 10^{-13}$ cm, $\mathcal{G} = 45$ MeV, $g = 0.211$. His results are (in the centre of mass system)

$$(3.4) \quad \sigma_{el}^{theo} = 18.0 \cdot 10^{-26} \text{ cm}^2, \quad (d\sigma/d\Omega)_{el}^{theo} = 4.50 \exp[-(\theta/0.356)^2],$$

where θ is the scattering angle of the neutron in radians. The total cross-section of the α -particle for 88 ± 2 MeV protons has been found to be $(19.6 \pm 0.7) \cdot 10^{-26} \text{ cm}^2$ (HILLMAN, STAHL and RAMSEY⁽³¹⁾). The ratio of the elastic to the total cross-section has been found by TANNENWALD⁽³²⁾ in a cloud chamber experiment to be 0.49 ± 0.07 , for a neutron beam of mean energy

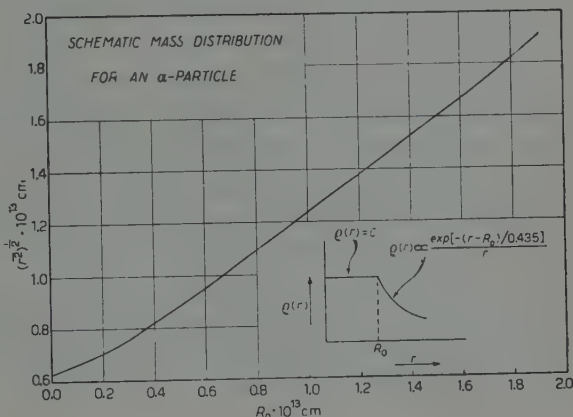


Fig. 3.1.

70 MeV and half width of 20 MeV. If we assume this figure to be valid at 88 MeV, the elastic cross-section becomes $\sigma_{el}^{(exp)} = 8.8 \cdot 10^{-26} \text{ cm}^2$. At such high energies, it is reasonable to assume that $\sigma \propto r_\alpha^2$. Hence to bring Heidmann's theoretical total cross-section into agreement with experiment it is necessary that r_α be reduced to

$$(3.5) \quad r_\alpha = 2.13 \cdot 10^{-13} \cdot (8.8/18.0)^{\frac{1}{2}} = 1.5 \cdot 10^{-13} \text{ cm}.$$

For the neutron beam mentioned above, TANNENWALD⁽³²⁾ gives $(d\sigma/d\Omega)_{el} \propto \exp[-5.0\theta^2]$ in the laboratory system. In the centre of mass system we therefore have approximately $(d\sigma/d\Omega)_{el}^{(exp)} \propto \exp[-(\theta/0.57)^2]$, where we have made a very small correction due to the geometrical factor which enters into the differential cross-section transformation formula. To obtain the corresponding result at 90 MeV, we assume that the denominator of θ , in the argument of the exponential, is $\theta_0 \propto (\hbar/r_\alpha)$.

Thus we have at 90 MeV, $\theta_0 = 0.57(70/90)^{\frac{1}{2}}$, and

$$(3.6) \quad (d\sigma/d\Omega)_{el}^{(exp)} \propto \exp[-(\theta/0.503)^2].$$

⁽³¹⁾ P. HILMAN, R. M. STAHL and N. F. RAMSEY: *Phys. Rev.*, **96**, 115 (1954).

⁽³²⁾ P. E. TANNENWALD: *Phys. Rev.*, **89**, 508 (1953).

This is much wider than the theoretical angular distribution (3.4). To bring the two figures into agreement, we must reduce HEIDMANN'S radius to

$$(3.7) \quad r_\alpha = 2.19 \cdot 10^{-13} \cdot (0.356/0.503) = 1.51 \cdot 10^{-13} \text{ cm } (*).$$

However, the results (3.5) and (3.7) are uncertain since the elastic nucleon scattering really measures not the extent r_α of the wavefunction but the extent of the nuclear force field. If the nuclear potential is of the form ⁽¹⁾

$$(3.8) \quad V_{ij} = \mathcal{G}_c f_c(r_{ij})(a_0 + a_\sigma \sigma_i \cdot \sigma_j + a_\tau \tau_i \cdot \tau_j + a_{\sigma\tau} \sigma_i \cdot \sigma_j \tau_i \cdot \tau_j) + \\ + \mathcal{G}_T f_T(r_{ij})(3\sigma_i \cdot \hat{r}_{ij} \sigma_j \cdot \hat{r}_{ij} - \sigma_i \cdot \sigma_j),$$

then the a_0 term gives rise to a field extending outside the wavefunction, while, owing to the saturation of spin and charge in the α -particle the other terms do not. The Serber potential contains a substantial a_0 term, while the symmetric potential ⁽¹⁾ does not. Thus the nucleon scattering leads to substantially different values of r_α according to the exchange character of the assumed nuclear potential, and our results (3.5) and (3.7) depend essentially on the use of the Serber potential. Although it would be necessary to recalculate the nucleon- α scattering with a potential of different exchange character before a definite conclusion could be drawn, it seems very probable that the symmetric potential would lead to a substantially bigger r_α than (3.5), (3.7).

In view of the uncertainty in r_α , we shall, where necessary, give results for both $r_\alpha = 1.5 \cdot 10^{-13}$ cm, $r_\alpha = 1.45 \cdot 4^{\frac{1}{3}} \cdot 10^{-13} = 2.3 \cdot 10^{-13}$ cm, the latter corresponding to the assumption of uniform density of nuclear matter. The corresponding values of R_0 are 0.9 and $1.75 \cdot 10^{-13}$ cm.

4. — Physical Criteria for the Validity of the α -Cluster Model.

4.1. — The criteria by which the model can be judged can be formulated in terms of three angular frequencies:

- (a) The frequency ω_{int} with which nucleons oscillate about the mass centres of their respective α -clusters.
- (b) The frequency ω_x with which an α -cluster exchanges nucleons with its neighbours.
- (c) The frequency ω_0 with which the whole nucleus vibrates or rotates.

(*) If we had taken into account the fact that the nuclear force field spreads outside r_α by an amount λ , we would have been led to reduce $(r_\alpha + \lambda)$ by the factor (1.5/2.19), and been led to a value of r_α smaller than (3.5) and (3.7).

If an α -cluster is to be a well-defined entity within a nucleus, it is necessary that each nucleon should carry out several complete oscillations within the cluster before escaping by leakage through the surrounding potential barriers. This requires that

$$(4.1) \quad \frac{1}{4}\omega_x \ll \omega_{\text{int}},$$

(since there are four nucleons in a cluster).

Since $\hbar\omega_{\text{int}} \cong 20$ MeV (section 4.2), a figure of the order of the α -particle binding energy, and since, moreover, the nucleon exchange energy between the α -clusters in a $4n$ nucleus is $\frac{1}{2}n\hbar\omega_x$, it follows from (4.1) that the α -particle model requires the internal binding energy of the α -clusters to be much larger than the nucleon exchange energy. (The estimate $\frac{1}{2}n\hbar\omega_x$ comes from a calculation of the energy of a lattice of α -clusters, provided one keeps only terms involving either no exchange or a single exchange of a pair of nucleons between neighbouring α -clusters. This approximation is valid for small exchange integrals. $\hbar\omega_x$ is identified with the exchange energy to which a single α -cluster contributes.)

The physical situation would be greatly simplified if, as was suggested by WHEELER⁽¹⁶⁾, the α -clusters were stable for periods long compared with the vibrational and rotational periods of the whole nucleus. This would be true if

$$(4.2) \quad \frac{1}{2}n\omega_x \ll \omega_0.$$

(The factor $\frac{1}{2}$ comes from the fact that 2 α -clusters take part in each exchange). The inequality (4.2) would imply that nucleon exchange forces only make a minor contribution to the interaction between α -clusters. This is because the total nuclear interaction between the clusters must be sufficiently large to counteract the **zero** point kinetic energy of the clusters and also the Coulomb repulsion. Thus we must have in the ground state

$$(4.3) \quad |\langle V \rangle| > f \cdot \frac{1}{4}\hbar\omega_0,$$

where $|\langle V \rangle|$ is the modulus of the total nuclear interaction energy which appeared in section 2. Combining (4.2) and (4.3) we have

$$(4.4) \quad |\langle V \rangle| \gg f \cdot \frac{1}{4} \cdot \frac{1}{2}n\hbar\omega_x.$$

Thus according to (4.2) the α - α forces would resemble the forces between oppositely charged ions, or the van der Waals forces between molecules, rather than the electron-exchange forces binding the hydrogen molecule. It also follows that the assumption of predominantly nucleon exchange forces is inconsistent with (4.2).

As the α -structure vibrates the internal state of the clusters will change. This occurs adiabatically if

$$(4.5) \quad \omega_{\text{int}} \gg \omega_0.$$

If (4.5) is violated the effective forces between the clusters may depend on the relative velocity.

We now proceed to estimate and compare the frequencies ω_{int} , ω_x , ω_0 .

4.2. *Magnitudes of ω_{int} , ω_x , ω_0 .* — The period τ of motion of a quantal system is given approximately by

$$(4.6) \quad \tau \cong \frac{2\pi\hbar}{D},$$

where D is the spacing of energy levels at the energy considered ⁽¹¹⁾. Since no excited bound state of the α -particle is known, the excitation energy of the first excited level is $(E^{(1)} - E_0^{(1)}) \cong 20$ MeV. The mode of excitation is essentially that of a single particle, so that the corresponding period is that for single nucleon vibration. We have therefore

$$(4.7) \quad \hbar\omega_{\text{int}} = \frac{2\pi\hbar}{\tau_{\text{int}}} \cong 20 \text{ MeV}.$$

(The periods of neutron and proton oscillation are practically the same since the difference of the triton and ^3He binding energies is only about 1 MeV).

To obtain an estimate of ω_x , we calculate the exchange energy of a nucleon vibrating in two one-dimensional potential troughs separated by a barrier. The troughs each represent α -clusters, and the barrier is due to their separation. According to (2.14) we have

$$(4.8) \quad \frac{1}{4} \hbar\omega_x \cong \frac{D}{2\pi} e^{-\sigma}.$$

The symbols D , G have the meaning assigned to them in section 2, except that $(4m)$ has to be replaced by m , and we put $[V(x) - E] \cong 20$ MeV, the difference between the α -particle and triton binding energies.

Equation (4.8) has a simple physical interpretation. The exchange frequency is equal to the frequency with which a nucleon vibrates inside its α -cluster, multiplied by a barrier penetration probability for each collision.

Putting the barrier length equal to $l \cong (s_\alpha - 2R_0)$, we have

$$G \cong l \{ (2m/\hbar^2) [V(x) - E] \}^{\frac{1}{2}} \cong l/1.0,$$

so that

$$(4.9) \quad \frac{1}{4} \hbar \omega_x \cong \frac{20}{2\pi} \exp[-0.21/1.0] = 2.6 \text{ MeV} \quad (\text{for } R_0 = 1.75 \cdot 10^{-13} \text{ cm}),$$

$$(4.10) \quad \cong \frac{20}{2\pi} \exp[-1.9/1.0] = 0.5 \text{ MeV} \quad (\text{for } R_0 = 0.9 \cdot 10^{-13} \text{ cm}).$$

This energy is extremely sensitive to the barrier length, and therefore to the radius of the α -particle. The estimate (4.9) must be regarded as only a first orientation because with such a small barrier the approximate formula (4.8) can hardly be accurate.

Estimates (4.9) and (4.10) are unrealistic because exchanges between α -clusters must always involve two nucleons. A complete α -particle cannot even approximately act as a deep potential trough for a single nucleon. This follows from the fact that the binding energy of an extra neutron to ^3He is 20 MeV, while the addition of a neutron to ^4He does not lead to a bound state. A more satisfactory estimate will be made in another paper after a detailed analysis of the α - α interaction. Another weakness of the argument leading to (4.9) and (4.10) is the use of the analogy to a one-dimensional problem.

An earlier estimate of ω_x has been made by WHEELER⁽¹⁶⁾ by calculating classically the time for a nucleon to diffuse through a half-wavelength of a collective nuclear motion. As WHEELER pointed out, the conditions for the classical considerations to be valid are not satisfied.

From the considerations of section 2, we have

$$(4.11) \quad \hbar \omega_0 \cong 4 \cdot 1.1 = 4.4 \text{ MeV}.$$

Since these estimates were made on the assumption of simple harmonic vibration, $\hbar \omega_0$ also represents the level spacing for this motion, so that ω_0 is the required frequency of α -cluster vibration.

4.3. *Frequency ratios.* - From (4.7) and (4.8) we have

$$(4.12) \quad \left(\frac{1}{4} \omega_x / \omega_{\text{int}} \right) \cong \frac{1}{2\pi} e^{-a} \cong 0.13 \quad (\text{for } R_0 = 1.75 \cdot 10^{-13} \text{ cm}),$$

$$(4.13) \quad \cong 0.024 \quad (\text{for } R_0 = 0.9 \cdot 10^{-13} \text{ cm}).$$

Thus the coherence requirement (4.1) is well satisfied, even for quite small barriers, although the accuracy of (4.12) must again be very poor.

From (4.9), (4.10) and (4.11) we see that the quasi-stability requirement (4.2) will be satisfied only for the large barrier, or smaller radius of the α -particle. This result is doubtful due to our crude estimate of ω_x .

From (4.7) and (4.11) we have

$$(4.14) \quad \omega_{\text{int}}/\omega_0 \cong 4.5.$$

This is not large, so that the effective interaction forces may well be velocity dependent.

4.4. Vibrational amplitude. — An estimate of the amplitudes of vibration of the α -clusters is required for two reasons. Firstly, our estimates of ω_x are on the basis of fixed mass centre separations s_α ; if the α -clusters are to maintain their identity, overlap must be small at all times during the motion. Secondly, the ratio $(\overline{\delta s_\alpha^2})^{1/2}/s_\alpha$ [where $(\overline{\delta s_\alpha^2})^{1/2}$ is the R.M.S. deviation of s_α] gives a criterion for the importance of rotation-vibration interaction.

INGLIS⁽¹⁸⁾ assumed an α - α potential varying parabolically with the deviation of the bond distance s_α from its mean value. According to his formulae, the simplest dilatational mode leads to $(\overline{\delta s_\alpha^2})^{1/2} = 0.76 \cdot 10^{-13}$ cm in ^{16}O , $1.32 \cdot 10^{-13}$ cm in ^{12}C and $2.71 \cdot 10^{-13}$ cm in ^8Be .

A rough criterion that overlap shall always be small is

$$(4.15) \quad R_0 < \frac{1}{2} [s_\alpha - (\overline{\delta s_\alpha^2})^{1/2}].$$

If we use (2.3), (2.4) and (2.5) for s_α in ^8Be , ^{12}C and ^{16}O respectively, then we obtain from (4.15), Fig. 3.1, and the values of $(\overline{\delta s_\alpha^2})^{1/2}$ just quoted, the upper limits $r_\alpha < 1.57, 1.77, 1.81 \cdot 10^{-13}$ cm in ^8Be , ^{12}C and ^{16}O respectively.

In the first excited state of the same dilatational mode, $(\overline{\delta s_\alpha^2})^{1/2}$ is increased by a factor $\sqrt{3}$ so that r_α must really be smaller than the values just derived. The presence of other vibrational modes also tends to reduce the upper limits for r_α . The self-consistency of the model therefore requires the α -particle to be substantially smaller than is indicated by the assumption of constant density of nuclear matter ($r_\alpha = 2.3 \cdot 10^{-13}$ cm). The results (3.5) and (3.7) would be quite satisfactory.

4.5. Comparison with atoms. — It is instructive to compare the criteria discussed above with the corresponding frequency ratios for the motion of electrons and nuclei in atoms and molecules. For the valence electrons, which are most affected by the presence of neighbouring atoms, we have (4.7)

$$(4.16) \quad \hbar\omega'_{\text{int}} \cong \bar{D}' \approx 5 \text{ eV}.$$

(We distinguish atomic quantities by primes.)

The potential barriers impeding the motion of electrons are of the order

of $[V'(x) - E] \approx 10$ eV high and $l' \approx 10^{-8}$ cm long. Thus

$$(4.17) \quad G' \cong l' \left\{ \frac{2m_e}{\hbar^2} [V'(x) - E] \right\}^{\frac{1}{2}} \cong 1.6,$$

where m_e is the electronic mass. Thus G' and G are of the same order of magnitude; [$G \cong 0.2$ to 1.9 ; see (4.8), (4.9)].

The vibrational frequencies encountered in molecules are of order of magnitude (see e.g. SCHIFF⁽³³⁾, p. 288)

$$(4.18) \quad \hbar\omega'_0 \approx \left(\frac{m_e}{Am} \right)^{\frac{1}{2}} \hbar\omega'_{\text{int}} = 0.023 A^{-\frac{1}{2}} \hbar\omega'_{\text{int}},$$

where A is the atomic weight of the nuclei.

From (4.7), (4.8) and (4.17) we see that

$$(4.19) \quad (\omega'_x/\omega'_{\text{int}}) \approx (\omega_x/\omega_{\text{int}}).$$

In both cases, this ratio varies very sensitively with G , G' .

From (4.8), (4.15), (4.16) and (4.17), we have

$$(4.20) \quad (\omega'_x/\omega'_0) \approx 1.4A^{\frac{1}{2}}.$$

On the other hand from (4.9), (4.10) and (4.11),

$$(4.21) \quad (\frac{1}{4}\omega_x/\omega_0) \approx 0.6 \text{ to } 0.1 \quad (*).$$

Therefore α -clusters are relatively much more stable than atoms. The difference between (4.20) and (4.21) is associated with the fact that electron exchange is very important in molecules, while, as discussed in 4.1, nucleon exchange between α -clusters may be negligible.

From (4.18) we have

$$(4.22) \quad (\omega'_{\text{int}}/\omega'_0) \approx 43A^{\frac{1}{2}},$$

while from (4.14) we had

$$(4.23) \quad (\omega_{\text{int}}/\omega_0) \cong 4.5.$$

Thus the relative motion of the nucleons in aggregates of α -clusters is much less adiabatic in character than the motion of electrons in molecules.

⁽³³⁾ L. I. SCHIFF: *Quantum Mechanics* (New York, 1949).

(*) Note that ω'_x refers to a single electron, while $\frac{1}{4}\omega_x$ refers to a single nucleon.

Lastly, we compare the vibrational amplitudes. In molecules, the R.M.S. amplitudes $(\delta s'^2)^{\frac{1}{2}}$ of nuclear vibration are of order of magnitude (see e.g. ⁽³³⁾, p. 290),

$$(4.24) \quad (\overline{\delta s'^2})^{\frac{1}{2}} \approx \left(\frac{m_0}{Am} \right)^{\frac{1}{2}} s' = 0.15 A^{-\frac{1}{2}} s',$$

where s' is the mean separation of the nuclei. The figures quoted in (4.7) show that e.g. for ¹⁶O

$$(4.25) \quad (\overline{\delta s_\alpha^2})^{\frac{1}{2}} \cong 0.24 s_\alpha.$$

The relative vibrational amplitude is larger still in ⁸Be and ¹²C, and of the order given in (4.25) for an infinite aggregate of α -clusters each of which has the kinetic energy given by (2.7). It follows that vibration-rotation interaction must be relatively much larger in aggregates of α -clusters than in molecules. The profound influence which the vibration of the α -cluster structure together with the indistinguishability of the α -clusters may have on the rotational level structure has been pointed out by TELLER and WHEELER ⁽²⁹⁾.

Another consequence of the large amplitudes in α -nuclei is that the anharmonicity of the α - α interaction may be important. In particular the probability density outside the range of the interaction between the α 's will not be negligible. On the one hand the wavefunction falls off as $\exp[-r^2/a^2]$ in the parabolic potential part of the classically forbidden region [$a \cong (\hbar^2/MD)^{\frac{1}{2}}$, where M is the mass of the oscillating particles and D the level spacing]; in α -nuclei we have, taking $D \cong 5$ MeV, $a \cong 1.4 \cdot 10^{-13}$ cm $\approx s_\alpha$; in molecules we have for nuclear vibrations with the aid of (4.16) and (4.18) $a \cong 2 \cdot 10^{-9} A^{-\frac{1}{2}}$ cm $\cong \cong 0.2 A^{-\frac{1}{2}} s'$. Thus, in α -nuclei the wavefunction will to a considerable extent leak out of the parabolic region, while in molecules this is not the case. On the other hand, the level spacing in molecules is small compared with the strength of the interaction, whereas for example in the ¹⁶O nucleus the first few excited states are at about 6 MeV, while the energy required for complete disintegration into four α -particles is 14.4 MeV. Outside the range of the interaction, the wave function falls off as $\exp[-r/a']/r$, where $a'^2 \cong 2M|E|/\hbar^2$ (here M is the mass of the particles and $|E|$ the absolute value of the total energy). In molecules, $a' \cong 3 A^{-\frac{1}{2}} \cdot 10^{-10}$ cm $\ll s'$ (for $E=2$ eV); in α -nuclei, $a' \cong 10^{-13}$ cm, not much smaller than s_α .

The appreciable leakage of the wavefunction outside the parabolic region in the α -model will increase in importance as the energy increases, and must be expected to seriously affect transition probabilities.

Acknowledgments.

I am grateful to Professor L. ROSENFELD for introducing me to this problem, and for his interest and encouragement during the progress of the work. Part of this work was done during the tenure of a maintenance grant from the Department of Scientific and Industrial Research.

RIASSUNTO (*)

Si stimano le energie cinetiche dei nuclei $A=4n$ (con $Z=N=2n$) fino ad $A=40$, sulla base del modello a particelle α . Si usa questo valore, assieme a note regolarità delle energie di legame e un valore stimato dell'energia coulombiana, per mostrare che, in questo modello, deve esistere un'ulteriore interazione addittiva tra particelle α vicine. Si danno nuovi criteri fisici per la stabilità delle particelle α e il grado di adiabaticità del loro moto; con l'ausilio della meccanica quantistica si valutano i parametri rilevanti. Si fa un confronto con l'analogo modello molecolare.

(*) Traduzione a cura della Redazione.

Studies on the α -Particle Model of Nuclei - II.

A. HERZENBERG

The Physical Laboratories - University of Manchester

(ricevuto il 14 Febbraio 1955)

Summary. — A formalism for the interaction of α -particles in nuclei is developed in an analogous way to the treatment of the interaction of atoms. The effective forces can be separated into a direct interaction (analogous to the long range coulomb interaction of ions), a nucleon exchange interaction (analogous to electron exchange), and a polarisation interaction (analogous to van der Waals forces). The direct interaction is additive and independent of velocity. The nucleon exchange interaction is velocity dependent and not additive. The polarisation interaction is velocity dependent, and whether or not it is additive is determined by the spin and charge dependence of the nuclear potential. The transition of the formalism to the usual treatment of atomic van der Waals forces is traced. Retardation effects due to the wave nature of the nuclear force field are unlikely to be present. It is shown that the asymptotic form of the forces between two α 's with a Yukawa π -meson potential contains exponential factors $\exp[-s/1.4]$ (direct), $\exp[-s/0.7]$ (polarisation), $\exp[-s/0.6]$ (exchange), where s is the separation of the mass centres in units of 10^{-13} cm. Arguments are given against the conclusion from earlier work that α -clusters in nuclear matter would dissolve.

1. — Introduction.

In a previous paper (HERZENBERG ⁽¹⁾, referred to as I) the possibility that nuclei might contain α -clusters as sub units, and the basic physical concepts behind this nuclear model were discussed. An analysis of the regularities in the binding energies of the $A = (4n)$ nuclei led to the conclusion that the α -cluster interactions must be additive with an expectation value of about 5.2 MeV per nearest neighbour pair (bond) of α -clusters.

(¹) A. HERZENBERG: *Nuovo Cimento*, **1**, 986 (1955).

The object of the present paper is to present the mathematical theory of the interaction of α -clusters, and to develop the analogy to the theory of inter-atomic forces. The discussion will be carried as far as is possible without introducing detailed assumptions about nuclear forces. A discussion of the magnitude of the effective forces between α -clusters will be given in another paper.

If one assumes the nuclear matter in a $4n$ nucleus to be divided into a number of α -clusters, the problem of the motion of the complete system is reduced to the determination of the relative motion of the α -clusters. This problem bears a strong resemblance to the theory of molecules (see I). However, there is a difference in that in molecules the existence of the heavy (compared with the electrons) nuclei enables one to solve the problem of electron motion in the adiabatic approximation (see e.g. ⁽²⁾) and to use the resultant electron energy as a potential determining, together with the Coulomb energy, the motion of the nuclei. On the other hand, in the interaction of α -clusters there is no immediate justification of the adiabatic approximation; but even if one ignores this difficulty and calculates the interaction energy of the α -clusters when their mass centres are confined to a domain small compared with their separation, it is not obvious that it is then permissible to treat the relative motion of the mass centres analogously to that of the nuclei in an atom, the place of the electron energy being taken by the calculated interaction energy.

The problem of two α -clusters has been treated by several authors who used different assumptions about the nuclear forces. For the present, only their methods will be outlined; a discussion of points intimately connected with the nuclear forces will be given in another paper.

The first attack on the problem of two α -clusters was made by HEISENBERG ⁽³⁾. HEISENBERG considered the wave function

$$(1.1) \quad \sum_P \pm \Psi_0^{(\alpha)}(1234) \Psi_0^{(\alpha)}(5678) \exp \left[-\eta \left(\frac{(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4)^2}{4} - \eta \left(\frac{\mathbf{r}_5 + \mathbf{r}_6 + \mathbf{r}_7 + \mathbf{r}_8}{4} - \mathbf{s} \right)^2 \right) \right],$$

which describes the two α -clusters as being confined to within a distance of about $\eta^{-\frac{1}{2}}$ from the points O, \mathbf{s} . (The symbol \sum_P implies a summation to antisymmetrise the wave function. $\Psi_0^{(\alpha)}(1234)$ is the ground state wavefunction of an α -particle containing nucleons 1, 2, 3, 4.) He calculated the interaction energy as the difference of the total energy and the energy of two isolated α -particles, each separately confined to within a distance $\eta^{-\frac{1}{2}}$ of a fixed point,

⁽²⁾ L. S. SCHIFF: *Quantum Mechanics* (New York, 1949).

⁽³⁾ W. HEISENBERG: *Zeits. f. Phys.*, **96**, 473 (1935).

as in (1.1). This estimate would be satisfactory in a lattice of a large number of α -clusters if one assumed the α - α interactions to be additive and ignored the collective rotation of the lattice. However, for a small number of α -clusters, the only reasonable procedure would be to use the calculated interaction energy as a potential in a wave equation for the relative motion of the mass centres. The validity of this step is doubtful due to the dependence of the interaction energy on the parameter η .

The problem was treated again by MARGENAU ⁽⁴⁾. His wavefunction was a totally antisymmetric sum of terms; in each of these the two α -particles were built up from single particle wave functions centred on two fixed points separated by a distance R . The interaction energy was again calculated by subtracting from the total energy the energy of two isolated α -particles, each with a wave function built up about a fixed point. Both HEISENBERG ⁽³⁾ and MARGENAU ⁽⁴⁾ calculated the polarisation and nucleon exchange interactions.

A third treatment of the problem was given by EDWARDS ⁽⁵⁾, who investigated the ground state of the ${}^8\text{Be}$ (2α) nucleus. EDWARDS used the wavefunction

$$(1.2) \quad \sum_P \pm r_g^2 \exp[-Cr_g^2 - A \sum_{i,j=1}^4 r_{ij}^2 - A \sum_{\alpha,\beta=5}^8 r_{\alpha\beta}^2] \cdot \text{spin and charge functions},$$

where A and C are parameters and

$$r_g^2 = [\tfrac{1}{8}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4) - \tfrac{1}{8}(\mathbf{r}_5 + \mathbf{r}_6 + \mathbf{r}_7 + \mathbf{r}_8)]^2. \quad (*)$$

EDWARDS pointed out that his wavefunction took account of some polarisation; however, the latter was only due to the distortion of an α -cluster from spherical to spheroidal shape, and did not include spin and charge dependent spatial correlations of nucleons in different α -clusters. Unfortunately the ground state of ${}^8\text{Be}$ is virtual (${}^8\text{Be} \rightarrow 2\alpha + 96 \text{ keV}$) so that the significance of a calculation with bound state boundary conditions is not clear. A more satisfactory treatment of the virtual levels of ${}^8\text{Be}$ would become possible if the problem could first be reduced to a two-particle wave equation containing effective forces of interaction of the α -clusters.

⁽⁴⁾ H. MARGENAU: *Phys. Rev.*, **59**, 37 (1941).

⁽⁵⁾ S. F. EDWARDS: *Proc. Camb. Phil. Soc.*, **48**, 652 (1952).

(*) EDWARDS' paper contains an error. His equation (4.17) for the expectation value of the potential energy can be checked by putting ($W=1$, $B=H=M=0$, $\kappa=0$) and comparing the result with the normalisation integral. This test is not satisfied. Dr. EDWARDS informs me that this is due to a misprint. The mean separations shown in his Fig. 1 are smaller by an order of magnitude than the expected nuclear radius at $A=8$.

A rigorous derivation of a wave-equation for the relative motion of a number of α -clusters was given by WHEELER⁽⁶⁾ (resonant group method). The treatment given in the present paper is based on this work. WHEELER considered the nucleon exchange processes between neighbouring aggregates of nucleons, this process being analogous to the exchange of electrons between atoms. Since the experimental data suggest an additive α - α interaction, it is of interest to extend the theory to polarisation forces, which are additive in the atomic case. We shall also show that, besides the exchange and polarisation forces, there is a third type of force, which we shall call the direct force. This is additive, and analogous to the coulomb interaction of ions.

The development of the theory will be made by two methods, both of which were suggested by WHEELER⁽⁶⁾ but not worked out by him in detail. In the first we shall use a trial wave function to describe the internal configuration of the α -clusters, and use Rayleigh's principle to determine the wave function describing their relative motion. This method leads to an equation, very similar to the Schrödinger equation, in which all three force terms appear. In the second method, nucleon exchange is completely neglected, and the wave function is developed in terms of the excited states of the internal motion of the α -clusters. This method leads to a form of perturbation theory. These methods are complementary. They both lead to the same expression for the direct force, while the first and second lead to lower and upper limits respectively for the polarisation force. The first is more convenient for the actual calculation of forces, while the second enables us to make the passage to the molecular van der Waals forces.

In section 2 we write down, assuming strong α -clustering, a trial wave-function for a $(4n)$ nucleus. The exclusion principle then enables us to show that the α -clusters must obey Bose-Einstein statistics.

In section 3 we shall use the wave-function given in section 2 to derive, with the aid of Rayleigh's principle, a wave equation for the relative motion of the α -clusters. The three different types of α - α -interaction will appear naturally during the course of this discussion. If the nucleon exchange process is neglected and the part of the wave-function describing the internal motion is assumed to be real, we are led to the adiabatic approximation familiar from molecular theory.

Section 4 gives the alternative mode of approach. We shall neglect the nucleon exchange between clusters, and assume that the nucleon exchange forces can be represented schematically by a hard sphere repulsion. The neglect of nucleon exchange enables us to treat the interaction potential between different α -clusters as a perturbation. The parts of the wave-functions containing

(6) J. A. WHEELER: *Phys. Rev.*, **52**, 1083 (1937).

excited states of the α -clusters will be eliminated. We shall then obtain a relative-motion wave equation in which the elimination of the excited states leads to polarisation forces.

In section 5 we shall estimate the velocity dependence of the polarisation forces, and compare the nuclear with the analogous molecular problem.

In section 6 we shall estimate the ranges of the α - α forces. The ranges of the direct and polarisation interaction can be obtained very simply from the general expressions given in sections 3 and 4. The magnitude of the nucleon exchange forces depends essentially on the overlap of the α -clusters; we shall therefore deduce the range of the exchange interaction from the asymptotic behaviour of the α -particle wave-function.

The additivity of the α - α forces will be discussed in section 7. The direct interaction can easily be shown to be additive; on the other hand, the exchange interaction is not additive, as we shall see from a consideration of the analogous electron exchange forces in molecules. The polarisation forces require a more complicated discussion, which leads to the conclusion that the additivity of these forces depends essentially on the details of the nuclear (two body) forces.

Section 8 contains a proof that the two alternative methods given in sections 3 and 4 provide means of bracketing the sum of the polarisation and direct forces.

Finally, in the Appendix, we shall discuss some arguments, which have been put forward by several authors, about the α -clustering process. The conclusion from the previous work on this subject was that although there must exist a tendency towards α -clustering in nuclei, this tendency is not very strong: α -clusters in nuclei were thought to dissolve rapidly. It seems to me that these arguments suggesting the solution of α -clusters in nuclear matter are unjustified.

The notation and nomenclature will be as in I.

2. - Form of the Wave Function.

Mathematically, the hypothesis that α -clusters form subunits of nuclear structure implies that a good approximation for the wave function of a $(4n)$ nucleus is ⁽⁶⁾

$$\begin{aligned}
 (2.1) \quad \Psi^{(n\alpha)}(\mathbf{r}_1 \dots \mathbf{r}_{4n}; \sigma_{1z} \dots \sigma_{4nz}; \tau_{1c} \dots \tau_{4nc}) = \\
 = \sum_P \pm \Phi([1,2,3,4], [5,6,7,8], \dots [(4n-3), (4n-2), (4n-1), (4n)]); \\
 X_{1234} \dots X_{4n-3,4n-2,4n-1,4n} \} \cdot F(X_{1234}, X_{5678}, \dots X_{4n-3,4n-2,4n-1,4n}).
 \end{aligned}$$

Here \mathbf{r}_i , σ_{iz} , $\tau_{i\zeta}$ are respectively the position, z co-ordinate of intrinsic spin, and ζ co-ordinate of isotopic spin of the i -th nucleon. The vector \mathbf{X}_{ijkl} is the mass centre of the particles (i, j, k, l) .

The function Φ contains the co-ordinates of all the nucleons in such a way that the latter are grouped into α -clusters. These α -clusters resemble free α -particles, but differ from them because of the exchange of nucleons and of their mutual polarisation. The polarisation is expressed by the dependence of Φ on the cluster position vectors \mathbf{X}_{ijkl} . The nucleon exchange is expressed by the summation over different groupings of the nucleons into α -clusters.

In order to take account of the exclusion principle, we require Φ to change sign when all the co-ordinates of any pair of particles in the same cluster are exchanged; moreover, in equation (2.1) we sum over all terms which may be derived from that shown by exchanging any number of pairs of particle co-ordinates, the two partners of a pair being taken from different clusters. The positive and negative signs respectively are assigned to terms obtained by exchanging an even or odd number of pairs in the term shown in (2.1).

When the distance between the cluster (1234) and the other clusters is much larger than the range of the nuclear forces, the co-ordinates of particles (1234) enter into Φ only in a factor $\Psi_0^{(\alpha)}(1234)$, the internal wave function of an α -particle in the ground state. Therefore when the clusters are so far apart that their nuclear interaction is small, Φ breaks up into a product of $\Psi_0^{(\alpha)}$ factors, and is invariant with respect to exchange of any pair of clusters. We restrict our considerations to functions which have this invariance for all separations of the clusters.

It now follows immediately from (2.1) that F must be invariant with respect to exchange of any pair of mass centre co-ordinates appearing in its argument. For the terms in (2.1) can be divided into $(4n!/[4!]^n n!)$ sets (where n is the number of α -clusters) in each of which the $4n$ particles are grouped into the same n clusters; the latter appear in the set in different permutations of order. Since a term is unchanged if the order of two clusters in Φ is reversed, F appears only in the symmetric form

$$(2.2) \quad \sum_{\mathbf{P}} + F(\mathbf{X}_{i_1}, \mathbf{X}_{i_2}, \dots, \mathbf{X}_{i_n}),$$

where $\mathbf{X}_{i_1}, \mathbf{X}_{i_2}, \dots$ are the mass centres of clusters i_1, i_2, \dots and $\sum_{\mathbf{P}}$ signifies summation over all functions obtained by writing the \mathbf{X}_i in any order. Thus the α -clusters obey Bose-Einstein statistics.

The clustering hypothesis on which (2.1) is based has been discussed by several authors. ROSENFELD (7) (Ch. XII) discusses the spatial correlations

(7) L. ROSENFELD: *Nuclear Forces* (Amsterdam, 1948).

of nucleons in the Fermi gas model. WHEELER ⁽⁶⁾ considered the relation between the α and independent particle models. GRÖNBLOM and MARSHAK ⁽⁸⁾ and WERGELAND ⁽⁹⁾ have discussed the lability of α -clusters in ^{12}C and ^8Be respectively. The conclusion of the latter authors that the α -clusters would « dissolve » appears to be unjustified. This work is commented on in detail in the Appendix.

3. — The Reduced Wave Equation. Variational Theory.

In this section we confine our attention to two α -clusters, and to the relative motion of the eight nucleons, so that the kinetic energy operator for the mass centre motion can be dropped from the hamiltonian. We use a wave function of the type (2.1) to obtain an approximate wave equation for the function $F(\mathbf{X})$ of the vector separation \mathbf{X} of the two mass centres. Our wave function is therefore

$$(3.1) \quad \Psi^{(2\alpha)} = \sum_{\mathbf{r}} \pm \Phi(1234, 5678; \mathbf{X}) F(\mathbf{X}),$$

where Φ and F have the symmetry properties specified in section 2.

We require that

$$(3.2) \quad \int \Phi^* \Phi dv_{\text{int}} = 1,$$

where the integration $\int dv_{\text{int}}$ extends over the spins, isotopic spins, and internal spatial co-ordinates of the two clusters. Equation (3.2) ensures that when the α -clusters are so far apart that the overlap of their wave functions is small, the probability density of the mass centre separation is given by $F^*(\mathbf{X}) F(\mathbf{X})$.

For the sake of simplicity, we require the wave function to vanish when the distance of any nucleon from the mass centre exceeds some large, fixed value L ; the eigen-values of the hamiltonian are therefore discrete. This requirement does not affect bound states. In scattering problems, the behaviour of the wave function is required at some distance L' of separation of the scatterer and scattered particle such that L' is much larger than the range of nuclear forces; by taking $L \gg L'$, the introduction of L does not affect the theory of the scattering, except for the introduction of discrete states.

Rayleigh's principle states that the quantity

$$(3.3) \quad \langle H \rangle = \int \Psi^* H \Psi / \int \Psi^* \Psi$$

⁽⁶⁾ B. GRÖNBLOM and R. MARSHAK: *Phys. Rev.*, **55**, 229 (1939).

⁽⁹⁾ H. WERGELAND: *Norske Vidensk. Selsk. Skrifter* No. 1 (1941).

is stationary when the function Ψ varies slightly about any of the exact eigenfunctions of H . (The integrations in (3.3) are over the spins, isotopic spins, and spatial co-ordinates of all the nucleons.) If Ψ_i is the i^{th} eigen function and E_i the associated eigenvalue of H then, putting $\Psi - \Psi_i = \delta\Psi_i$, we have $(\langle H \rangle - E_i)$ of second order in $\delta\Psi_i$. We therefore attempt to find the eigenvalues of H approximately by finding the stationary values of $\langle H \rangle$ with functions of the type (3.1). The analytical form of Φ is determined later in this section; for the moment we take this analytical form to be given. The function $F(X)$ is therefore determined by use of the variational principle

$$(3.4) \quad \delta \left(\int \Psi^* H \Psi / \int \Psi^* \Psi \right) = 0, \quad \text{or} \quad \delta \int \Psi^* (H - E') \Psi = 0.$$

The second statement in (3.4) follows from the first if we carry out the variation and put $E' = (\int \Psi^* H \Psi / \int \Psi^* \Psi)$, using the optimum Ψ .

The integrals in (3.4) are of two types:

$$(3.5) \quad \int \Psi^* H \Psi = \left(\int \Psi^* H \Psi \right)_D + \left(\int \Psi^* H \Psi \right)_X,$$

$$(3.6) \quad \int \Psi^* \Psi = \left(\int \Psi^* \Psi \right)_D + \left(\int \Psi^* \Psi \right)_X.$$

The quantities $()_D$ in (3.5) and (3.6), the « direct integrals », stand for

$$(3.7) \quad \left(\int \Psi^* H \Psi \right)_D = \frac{1}{2} \cdot \frac{8!}{4!4!} \int F^*(X) \Phi^*(1234, 5678; X) H \cdot F(X) \Phi(1234, 5678; X),$$

$$(3.8) \quad \left(\int \Psi^* \Psi \right)_D = \frac{1}{2} \cdot \frac{8!}{4!4!} \int F^*(X) \Phi^*(1234, 5678; X) \cdot F(X) \Phi(1234, 5678; X).$$

(We note that there are $8!/2 \cdot 4! \cdot 4!$ terms in the sum (3.1)). The terms $()_X$ in (3.5) and (3.6) contain the « exchange » integrals; e.g. $(\int \Psi^* \Psi)_X$ contains terms proportional to

$$(3.9) \quad \int F^*(X) \Phi^*(1234, 5678; X) F(\xi) \Phi(1235, 4678; \xi),$$

where

$$X = \frac{1}{4}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 - \mathbf{r}_5 - \mathbf{r}_6 - \mathbf{r}_7 - \mathbf{r}_8)$$

as before, and

$$(3.10) \quad \xi = \frac{1}{4}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 - \mathbf{r}_4 + \mathbf{r}_5 - \mathbf{r}_6 - \mathbf{r}_7 - \mathbf{r}_8). \quad (*)$$

(*) There is an ambiguity in the separation (3.5) due to the possible appearance of the spatial exchange operator P^s in H . When properly antisymmetrized wave functions are used, we have $P_{ij}^s = -P_{ij}^\sigma P_{ij}^\tau$, where P_{ij}^σ and P_{ij}^τ respectively are the mechanical spin and isotopic spin exchange operators for the pair of particles i, j . Hence there

The expression to be varied in accordance with (3.4) can be written in the form

$$(3.11) \quad \frac{1}{2} \cdot 8\bar{C}_4 \int \Psi^*(H - E')\Psi = \int F^*(\mathbf{X}) O(\mathbf{X}) F(\mathbf{X}) d\mathbf{X} - E' \int F^*(\mathbf{X}) F(\mathbf{X}) d\mathbf{X} + \\ + \int F^*(\mathbf{X}) P(\mathbf{X}, \boldsymbol{\xi}) F(\boldsymbol{\xi}) d\mathbf{X} d\boldsymbol{\xi},$$

where the integration has been carried out over all co-ordinates except the co-ordinate \mathbf{X} in the integrals $(\)_D$ and $(\mathbf{X}, \boldsymbol{\xi})$ in the integrals $(\)_X$. The first term in (3.11) comes from the direct integrals in (3.5), $O(\mathbf{X})$ being an operator on functions of \mathbf{X} . The second term comes from the direct integrals in (3.6). The third term comes from the exchange integrals in (3.5) and (3.6), and contains the energy E' . It can be shown, with the use of the antisymmetry of Ψ , and the fact that H is hermitian, that $P(\mathbf{X}, \boldsymbol{\xi}) = P^*(\boldsymbol{\xi}, \mathbf{X})$.

It is convenient to break up $O(\mathbf{X})$ into its component parts. We put

$$(3.12) \quad H = K + V,$$

where K is the kinetic and V the potential energy. Then the first term in (3.11) arises from

$$(3.13) \quad \int F^*(\mathbf{X}) \Phi^*(1234, 5678; \mathbf{X}) (K + V) F(\mathbf{X}) \Phi(1234, 5678; \mathbf{X}) d\mathbf{X} d\mathbf{r}_{\text{int}},$$

are two alternative ways of representing P_{ij}^x , and therefore of making the separation (3.5). We shall use the convention that all exchange operators in the nuclear potential are to be expressed in terms of P_{ij}^σ , P_{ij}^τ . Our convention leads to the result that the exchange integrals of the form

$$\int \Phi^*(1234, 5678; \mathbf{X}) V \Phi(1235, 4678; \boldsymbol{\xi}), \text{ etc.},$$

contribute only to spatial exchange integrals, i.e. the integrals $\int P(\mathbf{X}, \boldsymbol{\xi}) F(\boldsymbol{\xi}) d\boldsymbol{\xi}$ in our final equation (3.30). The direct integrals of the form

$$\int \Phi(1234, 5678; \mathbf{X}) V \Phi(1254, 5678; \mathbf{X})$$

contribute only to the direct and polarisation terms in (3.30). If P^x appeared explicitly in the nuclear potential, some direct terms would give spatial exchanges, and some exchange terms would contribute to spatial integrals of non-exchange form.

Our results are independent of the convention used to write down the nuclear potential. Whether one uses P^x or $-P^\sigma P^\tau$, the same integrals turn up in the calculation. As we shall see, the direct and polarisation integrals are distinguished by the fact that their range is determined essentially by the distance dependence of the nucleon-nucleon interaction, while the range of the exchange integrals is always limited by the spread of the α -cluster wave functions.

where the integration $\int dv_{\text{int}}$ is over all the « internal » co-ordinates left when \mathbf{X} is held constant. We make the further decomposition

$$(3.14) \quad K = K_x + K_{\text{int}} = -\frac{\hbar^2}{2\mu} \nabla_x^2 + K_{\text{int}},$$

where K_x is the relative, and K_{int} the internal kinetic energy of the α -clusters, and μ is their reduced mass. We have

$$(3.15) \quad -\frac{\hbar^2}{2\mu} \int F^* \Phi^* \nabla_x^2 (F \Phi) d\mathbf{X} dv_{\text{int}} = \\ = -\frac{\hbar^2}{2\mu} \left\{ \int F^*(\mathbf{X}) \nabla_x^2 F(\mathbf{X}) d\mathbf{X} - \int d\mathbf{X} F^*(\mathbf{X}) F(\mathbf{X}) \int \nabla_x \Phi^* \cdot \nabla_x \Phi dv_{\text{int}} \right\} - \\ - i\hbar \int F^* \mathbf{J}(\mathbf{X}) \cdot \nabla F d\mathbf{X} + \frac{\hbar}{2i} \int F^* [\nabla \cdot \mathbf{J}(\mathbf{X})] F d\mathbf{X},$$

where

$$(3.16) \quad \mathbf{J}(\mathbf{X}) = \frac{\hbar}{2\mu i} \int [\Phi^* (\nabla_x \Phi) - (\nabla_x \Phi^*) \Phi] dv_{\text{int}},$$

and where we have used (3.2) and the fact that Φ becomes independent of \mathbf{X} for large $|\mathbf{X}|$. The last three terms in (3.15) represent a contribution to the relative kinetic energy from the dependence of the « internal » wave function Φ on \mathbf{X} . We shall use the notation

$$(3.17) \quad \frac{\hbar^2}{2\mu} \int \nabla_x \Phi^* \cdot \nabla_x \Phi dv_{\text{int}} + \frac{\hbar}{2i} \nabla \cdot \mathbf{J}(\mathbf{X}) = \Delta K(\mathbf{X}).$$

The internal kinetic energy in (3.14) gives rise to a term

$$\int F^*(\mathbf{X}) F(\mathbf{X}) \bar{K}_{\text{int}}(\mathbf{X}) d\mathbf{X},$$

where

$$(3.18) \quad \bar{K}_{\text{int}}(\mathbf{X}) = \int \Phi^* K_{\text{int}} \Phi dv_{\text{int}}.$$

In (3.13) we also put

$$(3.19) \quad V = V_{\alpha\alpha} + V_{\text{int}},$$

where $V_{\alpha\alpha}$ is the interaction potential between nucleons in the two separate clusters, and V_{int} is the internal potential energy operator of the clusters

The potential V then gives rise to a term

$$(3.20) \quad \int F^*(\mathbf{X}) F(\mathbf{X}) [\bar{V}_{\alpha\alpha}(\mathbf{X}) + \bar{V}_{\text{int}}(\mathbf{X})] d\mathbf{X},$$

where

$$(3.21) \quad \bar{V}_{\alpha\alpha}(\mathbf{X}) = \int \Phi^* V_{\alpha\alpha} \Phi dv_{\text{int}}, \quad \bar{V}_{\text{int}}(\mathbf{X}) = \int \Phi^* V_{\text{int}} \Phi dv_{\text{int}}.$$

From equations (3.11) to (3.20), we have

$$(3.22) \quad \int F^*(\mathbf{X}) O(\mathbf{X}) F(\mathbf{X}) d\mathbf{X} = \\ = \int F^*(\mathbf{X}) \left[-\frac{\hbar^2}{2\mu} \nabla_{\mathbf{X}}^2 + \Delta K(\mathbf{X}) + \bar{V}_{\alpha\alpha}(\mathbf{X}) + \bar{K}_{\text{int}}(\mathbf{X}) + \bar{V}_{\text{int}}(\mathbf{X}) - i\hbar \mathbf{J}(\mathbf{X}) \cdot \nabla \right] F(\mathbf{X}) d\mathbf{X}.$$

It is easy to show that the operator $[(\hbar/2i)(\nabla \cdot \mathbf{J}) - i\hbar \mathbf{J} \cdot \nabla]$ is hermitian.

According to (3.4) we now have from (3.11)

$$(3.23) \quad \int \delta F^* O F d\mathbf{X} - E' \int \delta F^* \cdot F d\mathbf{X} + \int \delta F^*(\mathbf{X}) P(\mathbf{X}, \xi) F(\xi) d\mathbf{X} d\xi + \\ + \int \delta F O^* F^* d\mathbf{X} - E' \int \delta F \cdot F^* d\mathbf{X} + \int \delta F(\mathbf{X}) P^*(\mathbf{X}, \xi) F^*(\xi) d\mathbf{X} d\xi = 0,$$

where we have used $P(\mathbf{X}, \xi) = P^*(\xi, \mathbf{X})$. If (3.23) is valid for a variation δF , it must also be valid for $i\delta F$, so that F has to satisfy

$$(3.24) \quad -\frac{\hbar^2}{2\mu} \nabla^2 F(\mathbf{X}) + [\bar{V}_{\alpha\alpha}(\mathbf{X}) + \Delta K(\mathbf{X}) + \bar{K}_{\text{int}}(\mathbf{X}) + \bar{V}_{\text{int}}(\mathbf{X})] F(\mathbf{X}) + \\ + \int P(\mathbf{X}, \xi) F(\xi) d\xi - i\hbar \mathbf{J}(\mathbf{X}) \cdot \nabla F(\mathbf{X}) = E' F(\mathbf{X}).$$

It is convenient to rewrite the terms in equation (3.24) so as to make the effect of cluster polarisation explicit. For this purpose we define the polarisation contribution $\Delta\Phi$ to the internal cluster wavefunction by the equation

$$(3.25) \quad \Phi(\mathbf{X}) = \Psi_0^{(\alpha)}(1234) \Psi_0^{(\alpha)}(5678) + \Delta\Phi(\mathbf{X}),$$

where $\Psi_0^{(\alpha)}(1234)$ is the ground state internal wavefunction of an α -particle containing nucleons (1234). We now bring out the effect of polarisation on

$\bar{V}_{\alpha\alpha}$, \bar{K}_{int} , and \bar{V}_{int} by defining $\Delta\bar{V}_{\alpha\alpha}$ and $\Delta\bar{E}_{\text{int}}$ as follows:

$$(3.26) \quad \bar{V}_{\alpha\alpha}(\mathbf{X}) = \langle 00 | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle + \Delta V_{\alpha\alpha}(\mathbf{X}),$$

where

$$(3.27) \quad \langle 00 | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle = \int \Psi_0^{(\alpha)*}(1234) \Psi_0^{(\alpha)}(5678) V_{\alpha\alpha} \Psi_0^{(\alpha)}(1234) \Psi_0^{(\alpha)}(5678) dv_{\text{int}};$$

$$(3.28) \quad \bar{K}_{\text{int}}(\mathbf{X}) + \bar{V}_{\text{int}}(\mathbf{X}) = \int \Psi_0^{(\alpha)*} \Psi_0^{(\alpha)*} (K_{\text{int}} + \bar{V}_{\text{int}}) \Psi_0^{(\alpha)} \Psi_0^{(\alpha)} dv_{\text{int}} + \Delta E_{\text{int}}(\mathbf{X}).$$

$$= 2E_0^{(1)} + \Delta E_{\text{int}}(\mathbf{X}).$$

The polarisation contributions to the terms in (3.24) will be grouped together by defining the polarisation energy

$$(3.29) \quad V_{\text{pol}}(\mathbf{X}) = \Delta K(\mathbf{X}) + \Delta\bar{V}_{\alpha\alpha}(\mathbf{X}) + \Delta E_{\text{int}}(\mathbf{X}) - i\hbar \mathbf{J}(\mathbf{X}) \cdot \nabla.$$

The effect of polarisation on the integral term in (3.24) will not be made explicit.

With the aid of (3.25) to (3.28), we can now obtain the equation

$$(3.30) \quad -\frac{\hbar^2}{2\mu} \nabla_{\mathbf{X}}^2 F(\mathbf{X}) + \langle 00 | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle F(\mathbf{X}) + V_{\text{pol}}(\mathbf{X}) F(\mathbf{X}) +$$

$$+ \int P(\mathbf{X}, \boldsymbol{\xi}) F(\boldsymbol{\xi}) d\boldsymbol{\xi} = (E' - 2E_0^{(1)}) F(\mathbf{X}).$$

The first term is the relative kinetic energy. The second is the direct interaction. The third is the polarisation interaction due to the distortion of the α -cluster wave function; this term is velocity dependent since $-i\hbar \mathbf{J} \cdot \nabla = \mathbf{J} \cdot \mathbf{p}$. The fourth is the result of nucleon exchange and is also velocity dependent.

Henceforth we shall assume that the trial functions used for Φ are real, so that by (3.16) $\mathbf{J} = 0$, and $V_{\text{pol}}(\mathbf{X})$ is a static potential.

Equation (3.30) is not exactly a Schrödinger equation for F because $P(\mathbf{X}, \boldsymbol{\xi})$ contains the energy eigenvalue E' . However, the α -model can only be expected to be a good approximation when the relative energy $(E - 2E_0^{(1)})$ is small compared with the ground state energy $2E_0^{(1)}$ of the two α -clusters. We therefore put $2E_0^{(1)}$ for E' in $P(\mathbf{X}, \boldsymbol{\xi})$. With this approximation (3.30) now has the Schrödinger form. The integral term represents a velocity dependent interaction.

We have still to develop a criterion for the choice of the analytic form of Φ . According to Rayleigh's principle, the lowest energy eigenvalue E_0' of (3.30)

must lie above the true ground state energy E_0 of the system. We therefore adjust Φ so as to minimise E'_0 . In another paper, we shall show that the integral term is very probably repulsive and that its expectation value is probably negligible in low energy states. Therefore the attraction between the α -clusters must come from the terms $\langle 00 | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle + V_{\text{pol}}(\mathbf{X})$ in (3.30), so that E'_0 will be minimised by making this interaction as attractive as possible. Hence Φ is determined by

$$(3.31) \quad \delta\Phi[\langle 00 | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle + V_{\text{pol}}(\mathbf{X})] = 0,$$

or

$$\delta \int [\Phi(K_{\text{int}} + V_{\text{int}} + V_{\alpha\alpha})\Phi + \nabla_x \Phi \cdot \nabla_x \Phi] dv_{\text{int}} = 0,$$

together with (3.2) i.e.,

$$(3.32) \quad \delta \int \Phi^2 dv_{\text{int}} = 0.$$

If we multiply (3.32) by a Lagrangian multiplier E_{int} and subtract from (3.31), we have

$$(3.33) \quad \delta \int [\Phi(K_{\text{int}} + V_{\text{int}} + V_{\alpha\alpha} - E_{\text{int}})\Phi + \nabla_x \Phi \cdot \nabla_x \Phi] dv_{\text{int}} = 0.$$

The most general real variation of Φ is of the form

$$(3.34) \quad \delta\Phi = \delta(\mathbf{X} - \mathbf{X}_0) \delta\Phi_{\text{int}},$$

where $\delta\Phi_{\text{int}}$ is a real function of the internal co-ordinates alone. At the point $\mathbf{X} = \mathbf{X}_0$ we have $[\nabla_x \delta(\mathbf{X} - \mathbf{X}_0)]_{\mathbf{X} = \mathbf{X}_0} = 0$, so that the last term in (3.33) does not contribute. It follows that Φ must be a real solution of the equation

$$(3.35) \quad (K_{\text{int}} + V_{\text{int}} + V_{\alpha\alpha})\Phi = E_{\text{int}}\Phi.$$

In (3.35) the mass centre separation of the α -clusters is a constant, so that Φ has to satisfy the Schrödinger equation for the internal motion of the α -clusters in the adiabatic approximation and with neglect of nucleon exchange. This result appears as a consequence of the approximation (3.1), the requirement that Φ be real, and the consistent use of Rayleigh's principle.

If we multiply (3.35) on the left by Φ , integrate over the « internal » co-ordinates, and substitute in (3.30) for

$$[\langle 00 | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle + V_{\text{pol}}(\mathbf{X}) + 2E_0^{(0)}],$$

we obtain

$$(3.36) \quad -\frac{\hbar^2}{2\mu} \nabla_x^2 F + \Delta K(X)F + E_{\text{int}}(X)F + \int P(X, \xi) F(\xi) d\xi = E'F.$$

The energy of the internal motion therefore appears as a potential for the relative motion (*).

4. - The Reduced Wave Equation. Perturbation Theory.

In this section, we take as our starting point the assumption that the integral term in (3.30) is repulsive (a result suggested by the discussion of nucleon exchange forces which will be given in another paper) and that its expectation value is small. We also assume that the integral term can be represented schematically by a hard sphere potential

$$(4.1) \quad \begin{cases} V_x(X) = \infty & \text{for } |X| < X_0, \\ V_x(X) = 0 & \text{for } |X| > X_0, \end{cases}$$

where X_0 is of the order of magnitude of the α -particle diameter; also that, except for this potential, all nucleon exchange effects can be neglected. Physically, this assumption means that $\frac{1}{2}n\omega_x \ll \omega_0$ in the notation of I, section 4.

The hamiltonian is

$$(4.2) \quad H = K + V_{\alpha\alpha} + H_{\text{int}},$$

where K is the relative kinetic energy operator of the two clusters, $V_{\alpha\alpha}$ is the

(*) This discussion does not, of course, prove the validity of the adiabatic approximation when nucleon exchange is neglected. We have merely shown that in our approximation the adiabatic approximation gives the « best » form of Φ . The situation here is different from that in molecules, where simple estimates show that the periods of nuclear motion are much larger than those of electron motion (see I, section 4.5) and where it is therefore satisfactory, as a first approximation, to regard the nuclei fixed and to consider only the electron motion. In the interaction of α -clusters, the ratio of the periods of the mass centre motion to those of internal motion is much smaller than the corresponding ratio in molecules (see I, 4.5) and there is nothing to correspond to heavy nuclei in the molecules. It is here more satisfactory to start from the assumption of clustering implied in the wave function (2.1), and to obtain the adiabatic approximation as a consequence.

Equation (3.35) can be solved by a variational method. If this is done, the unperturbed function $\Psi_0^{(\alpha)}(1234)$ is itself to be regarded as a variational trial function which is adjusted to minimize the energy $E_0^{(1)}$.

interaction potential between them, and H_{int} is their internal energy. This decomposition of H is possible only when nucleon exchange is neglected.

The wave function can be written in the (exact) form

$$(4.3) \quad \Psi^{(2\alpha)} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} F_{ij}(\mathbf{X}) \Psi_i^{(\alpha)}(1234) \Psi_j^{(\alpha)}(5678).$$

Here the function $\Psi_i^{(\alpha)}$ is the internal wave function of the α -cluster consisting of nucleons (1234) in the i -th excited state. The summation extends over all excited states, including those in which the clusters become dissociated. The vector \mathbf{X} is the mass centre separation. The condition that $\Psi^{(2\alpha)}$ should be invariant when clusters (1234), (5678) are exchanged requires that

$$(4.4) \quad F_{ij}(\mathbf{X}) = F_{ji}(-\mathbf{X}).$$

We are interested in motions in which the two α -clusters are not appreciably perturbed from their ground state. Therefore the most important term in (4.2) is that with $i = j = 0$. In what follows, we write this term separately from the rest.

From (4.2) and (4.3), the Schrödinger equation

$$H\Psi = E\Psi,$$

becomes

$$(4.5) \quad [K + V_{\alpha\alpha} - (E - 2E_0^{(1)})] F_{00}(\mathbf{X}) \Psi_0^{(\alpha)}(1234) \Psi_0^{(\alpha)}(5678) + \\ + \sum'_{i,j} (K + E_i^{(1)} + E_j^{(1)} - E + V_{\alpha\alpha}) F_{ij}(\mathbf{X}) \Psi_i^{(\alpha)}(1234) \Psi_j^{(\alpha)}(5678) = 0,$$

where $E_i^{(1)}$ is the internal energy of an α -particle in the i -th excited state. The prime on \sum' indicates that the term with $i = j = 0$ is to be omitted in the summation.

If we multiply (4.5) on the left by $\Psi_i^{(\alpha)*}(1234) \Psi_j^{(\alpha)*}(5678)$ (where i, j run from zero to infinity) and integrate over the spins, isotopic spins and relative spatial co-ordinates of nucleons (1234) and (5678), we obtain the equations

$$(4.6) \quad \left[-\frac{\hbar^2}{2\mu} \nabla^2 + \langle 00 | V_{\alpha\alpha} | 00 \rangle - (E - 2E_0^{(1)}) \right] F_{00}(\mathbf{X}) = - \sum'_{i,j} \langle 00 | V_{\alpha\alpha} | ij \rangle F_{ij}(\mathbf{X}),$$

$$(4.7) \quad \left[-\frac{\hbar^2}{2\mu} \nabla^2 + \langle ij | V_{\alpha\alpha} | ij \rangle - (E - E_i^{(1)} - E_j^{(1)}) \right] F_{ij}(\mathbf{X}) = \\ = - \langle ij | V_{\alpha\alpha} | 00 \rangle F_{00}(\mathbf{X}) - \sum_{(k,l) \neq (i,j)} \langle ij | V_{\alpha\alpha} | kl \rangle F_{kl}(\mathbf{X}).$$

Here μ is the reduced mass of two α -clusters. We have put

$$(4.8) \quad \int \Psi_i^{(\alpha)*}(1234) \Psi_j^{(\alpha)*}(5678) V_{\alpha\alpha} \Psi_k^{(\alpha)}(1234) \Psi_l^{(\alpha)}(5678) dv_{\text{int}} = \langle ij | V_{\alpha\alpha} | kl \rangle,$$

where the integral $\int dv_{\text{int}}$ signifies integration over the spins, isotopic spins, and internal spatial co-ordinates of the clusters (1234), (5678).

We now assume that the state vector contains no incoming wave part in which the α -clusters are either excited or disrupted. Then we have from (4.7), to the first order in $V_{\alpha\alpha}$,

$$(4.9) \quad F_{ij}(\mathbf{X}) = -\frac{2\mu}{4\pi\hbar^2} \int \frac{\exp[ik_{ij}|\mathbf{X}-\boldsymbol{\xi}|]}{|\mathbf{X}-\boldsymbol{\xi}|} \langle ij | V_{\alpha\alpha}(\boldsymbol{\xi}) | 00 \rangle F_{00}(\boldsymbol{\xi}) d\boldsymbol{\xi},$$

if $E > (E_i^{(1)} + E_j^{(1)}),$

$$(4.10) \quad F_{ij}(\mathbf{X}) = -\frac{2\mu}{4\pi\hbar^2} \int \frac{\exp[-k_{ij}|\mathbf{X}-\boldsymbol{\xi}|]}{|\mathbf{X}-\boldsymbol{\xi}|} \langle ij | V_{\alpha\alpha}(\boldsymbol{\xi}) | 00 \rangle F_{00}(\boldsymbol{\xi}) d\boldsymbol{\xi},$$

if $E < E_i^{(1)} + E_j^{(1)},$

where

$$(4.11) \quad k_{ij} = |[2\mu(E - E_i^{(1)} - E_j^{(1)})/\hbar^2]^{\frac{1}{2}}|.$$

By use of the relation $\langle ij | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle = \langle ji | V_{\alpha\alpha}(-\mathbf{X}) | 00 \rangle$, which follows from (4.8), it is easily shown that (4.4) is satisfied by (4.9) and (4.10) if

$$F_{00}(\mathbf{X}) = F_{00}(-\mathbf{X}).$$

If we insert (4.9) and (4.10) in (4.6) we have, correct to the second order in $V_{\alpha\alpha}$,

$$(4.12) \quad \left\{ \begin{aligned} & -\frac{\hbar^2}{2\mu} \nabla^2 F_{00} + \langle 00 | V_{\alpha\alpha} | 00 \rangle F_{00}, \\ & -\frac{\mu}{2\pi\hbar^2} \sum'_{(E > E_i^{(1)} + E_j^{(1)})} \langle 00 | V_{\alpha\alpha}(\mathbf{X}) | ij \rangle \cdot \\ & \quad \cdot \int \frac{\exp[ik_{ij}|\mathbf{X}-\boldsymbol{\xi}|]}{|\mathbf{X}-\boldsymbol{\xi}|} \langle ij | V_{\alpha\alpha}(\boldsymbol{\xi}) | 00 \rangle F_{00}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \\ & -\frac{\mu}{2\pi\hbar^2} \sum'_{(E < E_i^{(1)} + E_j^{(1)})} \langle 00 | V_{\alpha\alpha}(\mathbf{X}) | ij \rangle \cdot \\ & \quad \cdot \int \frac{\exp[-k_{ij}|\mathbf{X}-\boldsymbol{\xi}|]}{|\mathbf{X}-\boldsymbol{\xi}|} \langle ij | V_{\alpha\alpha}(\boldsymbol{\xi}) | 00 \rangle F_{00}(\boldsymbol{\xi}) d\boldsymbol{\xi} = (E - 2E_0^{(1)}) F_{00}(\mathbf{X}). \end{aligned} \right.$$

Equation (4.12) contains only the function F_{00} . The term $\langle 00 | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle$

is the direct interaction. The two sums arise from the distortion of the α -clusters, and are the polarisation terms. The schematic exchange potential (4.1) has still to be included in (4.12).

Except that the symmetry condition (4.4) is not generally necessary, the analysis leading to (4.9), (4.10) and (4.12) applies just as well to the interaction of two atoms, or to the interaction of a single particle with a complicated nuclear or atomic system. In a collision process equation (4.12) describes the elastic scattering. In a stable system, (4.12) is sufficient to determine the eigenvalues of the energy correct to the second order in $V_{\alpha\alpha}$. The first sum in (4.21) is complex, and arises from inelastic scattering processes (4.9) which reduce the integrated probability density in the incident channel.

In the problem discussed in this paper, the total energy is always too small to bring about an inelastic collision between α -clusters. (Since the first excitation energy of an α -cluster must lie at about 20 MeV, inelastic collision cannot occur when the energy of relative motion is less than this). Therefore, for our purposes, we can write

$$(4.13) \quad \left\{ \begin{aligned} & -\frac{\hbar^2}{2\mu} \nabla^2 F_{00} + \langle 00 | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle F_{00}, \\ & -\frac{\mu}{2\pi\hbar^2} \sum'_{i,j} \langle 00 | V_{\alpha\alpha}(\mathbf{X}) | ij \rangle \cdot \\ & \cdot \int \frac{\exp[-k_{ij}|\mathbf{X}-\boldsymbol{\xi}|]}{|\mathbf{X}-\boldsymbol{\xi}|} \langle ij | V_{\alpha\alpha}(\boldsymbol{\xi}) | 00 \rangle F_{00}(\boldsymbol{\xi}) d\boldsymbol{\xi} = (E - 2E_0^{(1)}) F_{00}(\mathbf{X}). \end{aligned} \right.$$

The integral polarisation term in (4.13) depends on the form of F_{00} , and is therefore, in general, velocity dependent. This effect will be discussed further in section 5.

Comparison of (3.30) and (4.13) shows that the treatment by Rayleigh's principle (section 3) and the present perturbation theory lead to the same types of force. The direct and polarisation forces appear in both equations, and in both cases the polarisation forces turn out to be velocity dependent. The integral exchange term in (3.30) should be replaced in (4.13) by the phenomenological potential (4.1). The energies $E_0^{(1)}$ which appear on the right have slightly different meanings in (3.30) and (4.13). In (3.30), $E_0^{(1)}$ signifies the expectation value of the energy of an α -particle calculated with the trial function $\Psi_0^{(\alpha)}$ (3.25). In (4.13), $E_0^{(1)}$ is the exact ground state energy of the α -particle.

The formalism developed above requires only small changes to deal with assemblies of more than two α -clusters. Here we must first transform to the relative spatial co-ordinates. We replace the mass centre co-ordinates X_i ($i = 1 \dots n$) by

$$(4.16) \quad Y_i = \sum_{j=1}^n a_{ij} X_j,$$

where the transformation matrix satisfies the orthogonality relations

$$(4.17) \quad \sum_{k=1}^n a_{ik} a_{jk} = \sum_{k=1}^n a_{ki} a_{kj} = \delta_{ij}.$$

The new co-ordinate \mathbf{Y}_n is chosen to be proportional to the mass centre vector of the assembly; the rest are independent of the position of the mass centre. The transformation (4.16) and (4.17) leaves the kinetic energy operator unchanged in form:

$$(4.18) \quad K = -\frac{\hbar^2}{8m} \sum_{j=1}^n \nabla_j^2 = -\frac{\hbar^2}{8m} \sum_{j=1}^n \sum_{i=1}^3 \frac{\partial^2}{\partial X_{ji} \partial X_{ji}} = -\frac{\hbar^2}{8m} \sum_{j=1}^n \sum_{i=1}^3 \frac{\partial^2}{\partial Y_{ji} \partial Y_{ji}}.$$

Since we are not concerned with the motion of the mass centre, the corresponding term in (4.18) can be dropped. The term $-(\hbar^2/2\mu)\nabla^2$ in (4.6), (4.7) is replaced by the operator (4.18).

If we confine ourselves to energies too small to produce excited states, the Green's function $-(1/4\pi) \exp[-k_{ij}|\mathbf{X}-\boldsymbol{\xi}|]/|\mathbf{X}-\boldsymbol{\xi}|$ in (4.10) has to be replaced by

$$(4.19) \quad G_n(r; j) = -\frac{\exp[i\pi(3n-3)/4]}{2^{(3n-1)/2}\pi^{(3n-5)/2}} \cdot \left[\frac{8m}{\hbar^2} (E_j - E) \right]^{(3n-6)/4} \cdot \frac{H_{(3n-5)/2}^{(1)} \left\{ ir \left[\frac{8m}{\hbar^2} (E_j - E) \right]^{1/2} \right\}}{r^{(3n-5)/2}}$$

$$(4.20) \quad \sim \frac{-1}{2^{(3n-2)/2}\pi^{(3n-4)/2}} \left[\frac{8m}{\hbar^2} (E_j - E) \right]^{3n-6/4} \cdot \frac{1}{r^{(3n-4)/2}} \exp \left[-r \left[\frac{8m}{\hbar^2} (E_j - E) \right]^{1/2} \right],$$

for large r , where

$$(4.21) \quad r^2 = \sum_{k=1}^{n-1} (\mathbf{Y}_k - \boldsymbol{\eta}_k)^2,$$

the $\boldsymbol{\eta}_k$ being integration variables analogous to $\boldsymbol{\xi}$ in (4.10). In the equations (4.19), (4.20) we have used the suffix j as a collective label for a particular state of internal excitation of the α -clusters. The solution of the generalised equation (4.7) becomes, to the first order in $V_{\alpha\alpha}$,

$$(4.22) \quad F_j(\mathbf{Y}_1 \dots \mathbf{Y}_{n-1}) = \frac{8m}{\hbar^2} \int G_n(r; j) \langle j | V_{\alpha\alpha}(\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_{n-1}) | 0 \rangle \cdot F_0(\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_{n-1}) d\boldsymbol{\eta}_1 \dots d\boldsymbol{\eta}_{n-1},$$

for $j \neq 0$.

If we substitute (4.22) into the generalised equation (4.6), we obtain a generalised integral polarisation term. The connection of this term with the problem of additivity of forces of interaction between α -clusters will be discussed in section 7. If we wish to neglect the velocity dependence of the polarisation force, we put

$$\frac{8m}{\hbar^2} G_n(r; j) \rightarrow \frac{1}{(E - E_j)} \delta(\mathbf{Y}_1 - \boldsymbol{\eta}_1) \dots \delta(\mathbf{Y}_{n-1} - \boldsymbol{\eta}_{n-1}).$$

So far we have assumed that the nuclear forces can be derived from a static potential. If the nuclear forces are due to a meson field, we might expect retardation effects such as those discussed by CASIMIR and POLDER⁽¹⁰⁾ for the London-van der Waals forces between atoms. Let the meson field φ satisfy the equation.

$$(4.23) \quad \nabla^2 \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \left(\frac{Mc}{\hbar} \right)^2 \varphi = -\varrho,$$

where M is the meson mass, and ϱ the source density which we assume to vary periodically as $\exp[j\omega t]$. Retardation effects are due to the second term on the left of (4.23); this will be important only if

$$(4.24) \quad \left(\frac{Mc^2}{\hbar\omega} \right)^2 \lesssim 1.$$

For any particular term in the polarisation series in (4.13), we have $\hbar\omega \cong \cong (E_i^{(1)} + E_j^{(1)} - 2E_0^{(1)})$, while for the π meson $Mc^2 = 140$ MeV. Thus (4.24) will be satisfied only for $(E_i^{(1)} + E_j^{(1)} - 2E_0^{(1)}) \gtrsim 140$ MeV, so that the lower states with excitation energies ≈ 20 -30 MeV will not be affected.

5. - Velocity Dependence of the Forces.

We see from (3.30) and (4.13) that the direct interaction $\langle 00 | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle$ is just a static potential. The integral exchange term in (3.30) obviously depends on the rate of variation of the factor $F(\boldsymbol{\xi})$ in the integrand, and is therefore velocity dependent (see also section 6).

Last, we consider the polarisation forces. We start from (4.13). The effect of the velocity dependence is seen more clearly by calculating the expectation value of one of the integral terms in (4.13). We have [putting $\langle ij | V_{\alpha\alpha}(\boldsymbol{\xi}) | 00 \rangle$.

⁽¹⁰⁾ H. B. G. CASIMIR and D. POLDER: *Phys. Rev.*, **73**, 360 (1948).

$\cdot F(\xi) = g(\xi)$, and $\mathbf{X} = \mathbf{a} + \mathbf{b}$, $\xi = \mathbf{a} - \mathbf{b}$]

$$\begin{aligned}
 (5.1) \quad & - \frac{\mu}{2\pi\hbar^2} \int d\mathbf{X} d\xi g^*(\mathbf{X}) \frac{\exp[-\frac{k_{ij}}{|\mathbf{X}-\xi|}]}{|\mathbf{X}-\xi|} g(\xi) = \\
 & = - \frac{8\mu}{2\pi\hbar^2} \int d\mathbf{a} d\mathbf{b} g^*(\mathbf{a} + \mathbf{b}) \frac{\exp[-\frac{2k_{ij}|\mathbf{b}|}{2|\mathbf{b}|}]}{2|\mathbf{b}|} g(\mathbf{a} - \mathbf{b}) = \\
 & = - \frac{8\mu}{2\pi\hbar^2} \int d\mathbf{a} d\mathbf{b} [g^*(\mathbf{a}) + \mathbf{b} \cdot \nabla_{\mathbf{a}} g^*(\mathbf{a}) + \dots] \frac{\exp[-\frac{2k_{ij}|\mathbf{b}|}{2|\mathbf{b}|}]}{2|\mathbf{b}|} [g(\mathbf{a}) - \mathbf{b} \cdot \nabla_{\mathbf{a}} g(\mathbf{a}) + \dots] = \\
 & = - \frac{1}{|E - E_i^{(1)} - E_j^{(1)}|} \left[\int d\mathbf{a} g^*(\mathbf{a}) g(\mathbf{a}) - \frac{3\hbar^2}{4\mu |E - E_i^{(1)} - E_j^{(1)}|} \right. \\
 & \quad \left. \cdot \int d\mathbf{a} \nabla_{\mathbf{a}} g^*(\mathbf{a}) \cdot \nabla_{\mathbf{a}} g(\mathbf{a}) + \dots \right].
 \end{aligned}$$

If F_{00} is real, we have $F \nabla F = \frac{1}{2} \nabla F^2$, so that we can transform the second term in the bracket in (5.1) by partial integration:

$$\begin{aligned}
 (5.2) \quad & \int d\mathbf{X} \nabla g^*(\mathbf{X}) \cdot \nabla g(\mathbf{X}) = \int (\nabla F)^2 |\langle 00 | V_{\alpha\alpha}(\mathbf{X}) | ij \rangle|^2 d\mathbf{X} - \\
 & - \int F^2 \mathcal{R}(\langle ij | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle \nabla^2 \langle 00 | V_{\alpha\alpha}(\mathbf{X}) | ij \rangle) d\mathbf{X}.
 \end{aligned}$$

Thus the right hand side (5.1) becomes,

$$\begin{aligned}
 (5.3) \quad & - \frac{1}{|E - E_i^{(1)} - E_j^{(1)}|} \left\{ \int \left[|\langle 00 | V_{\alpha\alpha}(\mathbf{X}) | ij \rangle|^2 + \right. \right. \\
 & + \frac{3\hbar^2}{4\mu |E - E_i^{(1)} - E_j^{(1)}|} \mathcal{R}(\langle ij | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle \nabla^2 \langle 00 | V_{\alpha\alpha}(\mathbf{X}) | ij \rangle) \Big] F^2 d\mathbf{X} - \\
 & \left. - \frac{3\hbar^2}{4\mu |E - E_i^{(1)} - E_j^{(1)}|} \int |\langle 00 | V_{\alpha\alpha}(\mathbf{X}) | ij \rangle|^2 (\nabla F)^2 d\mathbf{X} + \dots \right\}.
 \end{aligned}$$

The first term in (5.3) is the contribution from the velocity independent approximation, and the second that from the velocity dependence in first approximation. The second term is always negative. The first is positive, for we have

$$\begin{aligned}
 |(3\hbar^2/4\mu |E - E_i^{(1)} - E_j^{(1)}|) \nabla^2 \langle ij | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle| & \approx |(3\hbar^2/8m \Delta E \lambda^2) \langle ij | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle| \lesssim \\
 & \lesssim 0.2 |\langle ij | V_{\alpha\alpha} | 00 \rangle|, \quad \text{for} \quad \Delta E = |E - E_i^{(1)} - E_j^{(1)}| \gtrsim 40 \text{ MeV}.
 \end{aligned}$$

Therefore the effect of velocity dependence is to reduce the polarisation energy.

This result may be given the following physical interpretation. The polar-

ization interaction is a consequence of the correlation due to $V_{\alpha\alpha}$ of the internal motions of the two clusters. When the period of the relative motion is not very much larger than the internal periods, the actual correlation of motions at any particular separation is not that leading to the maximum polarisation energy.

We estimate the velocity dependence from (5.3) by putting

$$(\nabla F)^2 \approx \frac{p^2}{\hbar^2} F^2, \quad \nabla^2 \langle 00 | V_{\alpha\alpha} | ij \rangle \approx \frac{1}{\lambda^2} \langle 00 | V_{\alpha\alpha} | ij \rangle,$$

and by replacing the denominators $|E - E_i^{(1)} - E_j^{(1)}|$ by a suitable average value ΔE . Then we have for the ratio R_v of the second to the first term in (5.3)

$$(5.4) \quad R_v \lesssim \left(\frac{3\hbar^2}{4\mu\Delta E} \cdot \frac{p^2}{\hbar^2} \right) / \left(1 + \frac{3\hbar^2}{8m\Delta E\lambda^2} \right) \cong \\ \cong \frac{3}{2} \cdot \frac{K}{\Delta E} / \left(1 + \frac{7.9}{\Delta E \text{ (MeV)}} \right) \approx 0.04K \text{ (MeV)},$$

where we have put $\lambda = 1.4 \cdot 10^{-13}$ cm, and $\Delta E \approx 40$ MeV. The connection with the considerations of (I, section 4) can be seen by putting

$$K \cong \frac{1}{2}(S + \frac{1}{2})\hbar\omega_0, \quad \Delta E \cong \hbar\omega_{\text{int}}.$$

We have

$$(5.5) \quad R_v \cong \frac{3}{4} \left(S + \frac{1}{2} \right) \frac{\omega_0}{\omega_{\text{int}}}.$$

It is interesting to estimate the velocity dependence of Van der Waals' forces between atoms. Let us put $\Delta E \approx 5$ eV, and $\mu = \frac{1}{2}mA$, where A is the atomic weight. The matrix elements of the interaction potential vary as R^{-3} , so that

$$\langle V \rangle \nabla^2 \langle V \rangle \approx (6/R^2) |\langle V \rangle|^2.$$

Hence we have for $R \approx 10^{-8}$ cm,

$$(5.6) \quad \frac{3\hbar^2}{4\Delta E} (\langle V \rangle \nabla^2 \langle V \rangle) / |\langle V \rangle|^2 \approx 10^{-2}/A;$$

accordingly the correction to the static part of the interaction is negligible. The ratio of the second to the first term in (5.3) becomes

$$(5.7) \quad R_v^{(\text{atoms})} \approx \frac{3\hbar^2}{4\mu\Delta E} \cdot \frac{p^2}{\hbar^2} \approx \frac{3}{2} \cdot \frac{K}{\Delta E} \approx (1840A)^{-\frac{1}{2}} = 0.023 A^{-\frac{1}{2}},$$

for nuclear vibrations (SCHIFF ⁽²⁾, p. 289).

From (5.7) we see that the velocity dependence in atoms is quite negligible. The Green's function

$$(\mu/2\pi\hbar^2)(\exp[\pm k_{ij}|\mathbf{X}-\boldsymbol{\xi}|]/|\mathbf{X}-\boldsymbol{\xi}|)$$

in (4.13) can therefore be replaced by

$$[\delta(\mathbf{X}-\boldsymbol{\xi})/E-E_i^{(0)}-E_j^{(0)}],$$

so that (4.13) reduces to the usual adiabatic perturbation theory formula for the Van der Waals force.

From (5.4), we see that the velocity dependent polarisation interaction between α -clusters is probably still negligible in the ground state ($K \cong 1$ MeV); it is unlikely to become appreciable at excitation energies smaller than about 20 MeV.

6. - Ranges of the Forces.

It can be seen from (3.27) that for a potential of sufficiently long range, the direct interaction in (3.30), (4.13) has the same range as the potential. It will be shown in another paper that this result is valid for the π -meson potential with a range $\lambda = 1.4 \cdot 10^{-13}$ cm.

The range of the polarisation interaction can be obtained most easily from the form given in (4.13). In the velocity independent approximation, the polarisation term has the form of the van der Waals interaction and is therefore roughly proportional to $\langle 00 | V_{\alpha\alpha}^2 | 00 \rangle$, in the notation of (3.27). Therefore, for a nuclear potential of exponential or Yukawa type, the polarisation range is half the nuclear force range.

We now discuss the range of the nucleon exchange interaction represented by the term $\int P(\mathbf{X}, \boldsymbol{\xi}) F(\boldsymbol{\xi}) d\boldsymbol{\xi}$ in (3.30). This term has its origin in integrals of the form (3.9). We consider only the contribution from the undistorted α -particle wave function. A common feature of all the terms making up the kernel $P(\mathbf{X}, \boldsymbol{\xi})$ is that their range in both \mathbf{X} and $\boldsymbol{\xi}$ is of the order of the radius of the α -particle. We attempt to estimate the distance behaviour of the nucleon exchange interaction by discussing the term of longest range in $P(\mathbf{X}, \boldsymbol{\xi})$. This comes from the integral

$$(6.1) \quad K(\mathbf{X}, \boldsymbol{\xi}) = \int \Psi_0^{(\alpha)*}(1234) \Psi_0^{(\alpha)*}(5678) \Psi_0^{(\alpha)}(1235) \Psi_0^{(\alpha)}(4678) d\mathbf{r}_{\text{int}},$$

where we integrate over all spatial co-ordinates, holding \mathbf{X} and $\boldsymbol{\xi}$ (3.10) constant, and sum over the spin and isotopic spin co-ordinates. \mathbf{X} and $\boldsymbol{\xi}$ are defined

by the equations

$$(6.2) \quad \mathbf{X} = \frac{1}{4}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 - \mathbf{r}_5 - \mathbf{r}_6 - \mathbf{r}_7 - \mathbf{r}_8),$$

$$(6.3) \quad \boldsymbol{\xi} = \frac{1}{4}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 - \mathbf{r}_4 + \mathbf{r}_5 - \mathbf{r}_6 - \mathbf{r}_7 - \mathbf{r}_8).$$

It is convenient to make the following transformation of co-ordinates:

$$(6.4) \quad \left\{ \begin{array}{l} \mathbf{Y}_1 = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2), \\ \mathbf{Y}_2 = \frac{1}{\sqrt{6}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3), \\ \mathbf{Y}_3 = \frac{1}{\sqrt{2}}(\mathbf{r}_6 - \mathbf{r}_7), \\ \mathbf{Y}_4 = \frac{1}{\sqrt{6}}(\mathbf{r}_6 + \mathbf{r}_7 - 2\mathbf{r}_8), \\ \mathbf{u} = \left[\frac{\sqrt{2}}{3}\mathbf{Y}_5 = \frac{1}{2} \left[\frac{1}{3}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) + \frac{1}{3}(\mathbf{r}_6 + \mathbf{r}_7 + \mathbf{r}_8) - \mathbf{r}_4 - \mathbf{r}_5 \right] \right], \end{array} \right.$$

and then

$$(6.5) \quad \frac{2}{3}(\mathbf{X} - 2\boldsymbol{\xi}) = \boldsymbol{\delta}.$$

The integral (6.1) now becomes, with an obvious change of notation,

$$(6.6) \quad K(\mathbf{X}, \boldsymbol{\delta}) = \sum_{\text{spins}} \int \Psi_0^{(\alpha)*}[\mathbf{Y}_1, \mathbf{Y}_2; (-\boldsymbol{\delta} - \mathbf{u})] \Psi_0^{(\alpha)*}[\mathbf{Y}_3, \mathbf{Y}_4; (\boldsymbol{\delta} - \mathbf{u})] \cdot \\ \cdot \Psi_0^{(\alpha)}[\mathbf{Y}_3, \mathbf{Y}_4; (\mathbf{X} + \frac{1}{2}\boldsymbol{\delta} - \mathbf{u})] \Psi_0^{(\alpha)}[\mathbf{Y}_1, \mathbf{Y}_2; (-\mathbf{X} - \frac{1}{2}\boldsymbol{\delta} - \mathbf{u})] d\mathbf{Y}_1 d\mathbf{Y}_2 d\mathbf{Y}_3 d\mathbf{Y}_4 d\mathbf{u}.$$

The quantities in round brackets inside square brackets in (6.6) are the vector separations of the fourth particle from the mass centre of the other three.

We first consider (6.6) when $|\mathbf{X}| > |\boldsymbol{\delta}| \gg \mu$, where μ is the range parameter in the asymptotic α -particle wave function when one nucleon is separated from the other three. It can easily be shown that

$$\Psi_0^{(\alpha)}(\mathbf{Y}_1, \mathbf{Y}_2; \mathbf{s}) \sim \Psi_0^{(t)}(\mathbf{Y}_1, \mathbf{Y}_2) \exp[-s/\mu]/s,$$

where

$$\mu = \left(\frac{3m|E_0^{(1)} - E_0^{(t)}|}{2\hbar^2} \right)^{\frac{1}{2}};$$

here $(\Psi_0^{(\alpha)}, \Psi_0^{(t)})$ and $(E_0^{(1)}, E_0^{(t)})$ are the ground state wavefunctions and energies of the α -particle and triton respectively, s is the separation of the odd nucleon and the mass centre of the other three, and m is the nucleon mass. (The parameter μ here governs the fall off of the wavefunction as a function of the separation of odd nucleon and triton; the parameter μ in I referred to the distance of the odd nucleon from the mass centre of the α -particle.) Under these conditions we also have $|\mathbf{X} + \frac{1}{2}\boldsymbol{\delta}| \gg \mu$. The asymptotic form of $\Psi_0^{(\alpha)}(\mathbf{Y}_1, \mathbf{Y}_2; (\mathbf{s}))$ will become valid when $|\mathbf{s}| \gtrsim \lambda$, since $\mathbf{Y}_1, \mathbf{Y}_2$ will generally be no larger than the dimensions of an α -particle. (Here λ is the range of the nuclear forces). Since we have $\lambda \cong \mu$, the functions $\Psi_0^{(\alpha)}$ in the integrand of (6.6) may, under the restrictions stipulated above, be replaced by their asymptotic forms, unless \mathbf{u} lies near to the points $\mathbf{u} = \pm \boldsymbol{\delta}$ and $\pm (\mathbf{X} + \frac{1}{2}\boldsymbol{\delta})$ which are marked A, B, C, D in Fig. 6.1. If we neglect this restriction for the moment, we have

$$(6.7) \quad K(\mathbf{X}, \boldsymbol{\delta}) \propto \sum_{\text{spins}} \int |\Psi_0^{(t)}(\mathbf{Y}_1, \mathbf{Y}_2)|^2 |\Psi_0^{(t)}(\mathbf{Y}_3, \mathbf{Y}_4)|^2 d\mathbf{Y}_1 d\mathbf{Y}_2 d\mathbf{Y}_3 d\mathbf{Y}_4 \cdot \\ \cdot \int \frac{\exp[-(|\boldsymbol{\delta} + \mathbf{u}| + |\boldsymbol{\delta} - \mathbf{u}| + |\mathbf{X} + \frac{1}{2}\boldsymbol{\delta} + \mathbf{u}| + |\mathbf{X} + \frac{1}{2}\boldsymbol{\delta} - \mathbf{u}|)/\mu]}{|\boldsymbol{\delta} + \mathbf{u}| |\boldsymbol{\delta} - \mathbf{u}| |\mathbf{X} + \frac{1}{2}\boldsymbol{\delta} + \mathbf{u}| |\mathbf{X} + \frac{1}{2}\boldsymbol{\delta} - \mathbf{u}|} d\mathbf{u}.$$

The factor

$$\exp[-(|\mathbf{X} + \frac{1}{2}\boldsymbol{\delta} + \mathbf{u}| + |\mathbf{X} + \frac{1}{2}\boldsymbol{\delta} - \mathbf{u}|)/\mu]$$

is equal to

$$\exp[-2|\mathbf{X} + \frac{1}{2}\boldsymbol{\delta}|/\mu]$$

on the line AB, and falls off rapidly to the sides and beyond the ends. The factor $\exp[-(|\boldsymbol{\delta} + \mathbf{u}| + |\boldsymbol{\delta} - \mathbf{u}|)/\mu]$ behaves similarly along the line CD, where it is equal to $\exp[-2|\boldsymbol{\delta}|/\mu]$.

If, for example, \mathbf{u} is near the point $(\mathbf{X} + \frac{1}{2}\boldsymbol{\delta})$, the substitution of the asymptotic form in (6.7) is not justified. In this region the third factor in (6.6) cannot be replaced by its asymptotic form; this factor will here be a continuous bounded function of $(\mathbf{X} + \boldsymbol{\delta} - \mathbf{u})$, instead of showing the $(1/|\mathbf{X} + \frac{1}{2}\boldsymbol{\delta} - \mathbf{u}|)$ singularity of the asymptotic form. However, the fourth factor here contains $\exp[-2|\mathbf{X} + \frac{1}{2}\boldsymbol{\delta}|/\mu]$, so that the product of the third and fourth factors is continuous and contains a factor $\exp[-2|\mathbf{X} + \frac{1}{2}\boldsymbol{\delta}|/\mu]$ along the whole line AB, including the end points. A similar argument can be applied to the points C and D, so that we are now justified in using (6.7) to represent the behaviour of (6.6) with respect to \mathbf{X} and $\boldsymbol{\delta}$ whenever $|\mathbf{X}| > |\boldsymbol{\delta}| \gg \mu$.

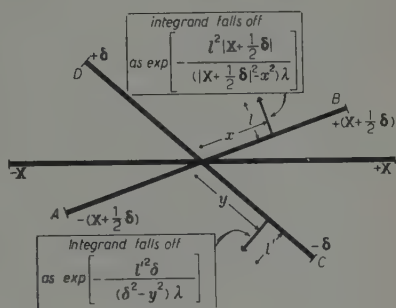


Fig. 6.1.

The factors

$$\exp[-(|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta} + \mathbf{u}| + |\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta} - \mathbf{u}|)/\mu], \quad \exp[-(|\boldsymbol{\delta} + \mathbf{u}| + |\boldsymbol{\delta} - \mathbf{u}|)/\mu]$$

are significant on «ridges» along the lines AB and CD respectively. The main contribution to the integral comes from the region where these ridges overlap, and where we can replace the exponentials by

$$\exp[-2(|\boldsymbol{\delta}| + |\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta}|)/\mu].$$

The integral therefore behaves as

$$\exp[-2(|\boldsymbol{\delta}| + |\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta}|)/\mu].$$

Since we have

$$(6.8) \quad (|\mathbf{X}| + \tfrac{1}{2}|\boldsymbol{\delta}|) \leq (|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta}| + |\boldsymbol{\delta}|) \leq (|\mathbf{X}| + \tfrac{3}{2}|\boldsymbol{\delta}|),$$

(6.6) falls off exponentially with \mathbf{X} and $\boldsymbol{\delta}$ separately, with a range $\approx \tfrac{1}{2}\mu$.

We shall later have to integrate $K(\mathbf{X}, \boldsymbol{\delta})$ over $\boldsymbol{\delta}$. Hence in calculating $K(\mathbf{X}, \boldsymbol{\delta})$, we can confine our attention to the region in which $|\boldsymbol{\delta}|$ is not much larger than μ . We therefore return to (6.6), and replace only the last two factors by their asymptotic form. This is justified if $|\mathbf{X}| \gg \mu$, since the condition $|\boldsymbol{\delta}| \lesssim \mu$, and the behaviour of the first two factors then ensure that $|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta} - \mathbf{u}| \gg \mu$, $|\mathbf{X} - \tfrac{1}{2}\boldsymbol{\delta} - \mathbf{u}| \gg \mu$ in the significant domain. We therefore have in place of (6.6)

$$(6.9) \quad K(\mathbf{X}, \boldsymbol{\delta}) \propto \sum_{\text{spins}} \int \Psi_0^{(\alpha)*}[\mathbf{Y}_1, \mathbf{Y}_2; (\boldsymbol{\delta} - \mathbf{u})] \Psi_0^{(\alpha)*}[\mathbf{Y}_3, \mathbf{Y}_4; (\boldsymbol{\delta} - \mathbf{u})] \cdot \\ \cdot \Psi_0^{(t)}(\mathbf{Y}_3, \mathbf{Y}_4) \Psi_0^{(t)}(\mathbf{Y}_1, \mathbf{Y}_2) d\mathbf{Y}_1 d\mathbf{Y}_2 d\mathbf{Y}_3 d\mathbf{Y}_4 \cdot \\ \cdot \int \frac{\exp[-(|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta} - \mathbf{u}| + |\mathbf{X} - \tfrac{1}{2}\boldsymbol{\delta} - \mathbf{u}|)/\mu]}{|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta} - \mathbf{u}| |\mathbf{X} - \tfrac{1}{2}\boldsymbol{\delta} - \mathbf{u}|} d\mathbf{u}.$$

We now have, in the neighbourhood of AB ,

$$(6.10) \quad \frac{\exp[-(|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta} - \mathbf{u}| + |\mathbf{X} - \tfrac{1}{2}\boldsymbol{\delta} - \mathbf{u}|)/\mu]}{|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta} - \mathbf{u}| |\mathbf{X} - \tfrac{1}{2}\boldsymbol{\delta} - \mathbf{u}|} \cong \frac{\exp[-2|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta}|/\mu]}{|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta}|^2} \cdot \\ \cdot \exp\left[\frac{-l^2|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta}|}{(|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta}|^2 - x^2)\mu}\right],$$

where l is the perpendicular distance of the point \mathbf{u} from the line AB , and x is the distance from the origin of the foot of the perpendicular from \mathbf{u} to AB . Substituting (6.10) in (6.9) we have

$$(6.11) \quad K(\mathbf{X}, \boldsymbol{\delta}) \propto \frac{\exp[-2|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta}|/\mu]}{|\mathbf{X} + \tfrac{1}{2}\boldsymbol{\delta}|^2} g(\boldsymbol{\delta}, \mathbf{X}),$$

where

$$(6.12) \quad g(\delta, \mathbf{X}) = \sum_{\text{spins}_s} \int \Psi_0^{(\alpha)*}[\mathbf{Y}_1, \mathbf{Y}_2; -(\delta - \mathbf{u})] \Psi_0^{(\alpha)*}[\mathbf{Y}_3, \mathbf{Y}_4; (\delta - \mathbf{u})] \cdot \\ \cdot \Psi_0^{(t)}(\mathbf{Y}_3, \mathbf{Y}_4) \Psi_0^{(t)}(\mathbf{Y}_1, \mathbf{Y}_2) d\mathbf{Y}_1 d\mathbf{Y}_2 d\mathbf{Y}_3 d\mathbf{Y}_4 \cdot \int \exp \left[\frac{-l^2 \mathbf{X} + \frac{1}{2}\delta}{\mu(|\mathbf{X} + \frac{1}{2}\delta|^2 - x^2)} \right] d\mathbf{u}.$$

The exponential factor in the integrand of (6.12) is significantly different from zero within a prolate spheroid with axis AB and with a radius $\cong [\mu(|\mathbf{X} + \frac{1}{2}\delta|^2 - x^2)/|\mathbf{X} + \frac{1}{2}\delta|]^{\frac{1}{2}}$ at distance x from the origin; the radius at the origin is $\cong (\mu|\mathbf{X} + \frac{1}{2}\delta|)^{\frac{1}{2}}$. Since the product $\Psi_0^{(\alpha)*} \Psi_0^{(\alpha)*} \Psi_0^{(t)} \Psi_0^{(t)}$ in (6.12) falls off as roughly $\exp[-2|\mathbf{u}|/\mu]$ when $|\mathbf{u}| \gtrsim (|\delta|, \mu)$, the value of (6.12) will be determined entirely by this product if the condition $|\mathbf{X} + \frac{1}{2}\delta| \gg \mu$ is supplemented by $(\mu|\mathbf{X} + \frac{1}{2}\delta|)^{\frac{1}{2}} \gg |\delta|$. Thus, under these conditions, we have $g(\delta, \mathbf{X}) \cong g'(|\delta|)$, where

$$(6.13) \quad g'(|\delta|) = \sum_{\text{spins}_s} \int \Psi_0^{(\alpha)*} \Psi_0^{(\alpha)*} \Psi_0^{(t)} \Psi_0^{(t)} d\mathbf{Y}_1 d\mathbf{Y}_2 d\mathbf{Y}_3 d\mathbf{Y}_4 d\mathbf{u}.$$

An approximate formula for $g'(|\delta|)$ can be obtained by replacing the factors $\Psi_0^{(\alpha)*}$ in (6.13) by their asymptotic forms. The spin summation and the integration $d\mathbf{Y}_1 \dots d\mathbf{Y}_4$ gives a factor one, so that

$$(6.14) \quad g'(|\delta|) \cong \int \frac{\exp[-(|\delta + \mathbf{u}| + |\delta - \mathbf{u}|)/\mu]}{|\delta + \mathbf{u}| |\delta - \mathbf{u}|} d\mathbf{u} = 2\pi\mu \exp[-2|\delta|/\mu].$$

[The integral in (6.14) has been evaluated by introducing prolate spheroidal co-ordinates with the foci at the points $\mathbf{u} = \pm \delta$].

We now return to (6.1), and express $K(\mathbf{X}, \delta)$ in terms of an equivalent velocity dependent operator $V(\mathbf{X}, \mathbf{p})$. We define $V(\mathbf{X}, \mathbf{p})$ by the relation (7)

$$(6.15) \quad V(\mathbf{X}, \mathbf{p}) F(\mathbf{X}) = \int K(\mathbf{X}, \xi) F(\xi) d\xi,$$

which is to be valid for any wave function F . We then have

$$(6.16) \quad V(\mathbf{X}, \mathbf{p}) = \int K(\mathbf{X}, \xi) \exp[(i/\hbar)(\xi - \mathbf{X}) \cdot \mathbf{p}] d\xi.$$

[In (6.15) and (6.16) \mathbf{p} stands for $(\hbar/i)\nabla$]. The result (6.16) may be verified by introducing the Fourier transform of $F(\mathbf{X})$ into (6.15).

For sufficiently large \mathbf{X} , we can replace $K(\mathbf{X}, \xi)$ in (6.16) by (6.11). We

make the further approximation of replacing $g(\mathbf{X}, \boldsymbol{\delta})$ by $g'(|\boldsymbol{\delta}|)$ given by (6.14). We then have

$$(6.17) \quad V(\mathbf{X}, \mathbf{p}) \propto \frac{1}{X^2} \int \exp \left[-\frac{2(|\mathbf{X} + \frac{1}{2}\boldsymbol{\delta}| + |\boldsymbol{\delta}|)}{\mu} - \frac{i}{\hbar} \left(\frac{1}{2}\mathbf{X} + \frac{3}{4}\boldsymbol{\delta} \right) \cdot \mathbf{p} \right] d\boldsymbol{\delta}$$

where we have used (6.5).

Since we have $(|\mathbf{X} + \frac{1}{2}\boldsymbol{\delta}| + |\boldsymbol{\delta}|) \geq (|\mathbf{X} + \frac{1}{2}\boldsymbol{\delta}|)$, the integral (6.17) converges if $(1/\mu) > (3|\mathbf{p}|/4\hbar)$. Putting $|\mathbf{p}| = \sqrt{8mK}$, where K is the relative kinetic energy of the two α -clusters, we have $(3\mu|\mathbf{p}|/4\hbar) \cong \frac{1}{3}$, (with $\mu = 1.2 \cdot 10^{-13}$ cm, $K = 1.1$ MeV). The integral (6.17) therefore converges in the ground state, and will also do so in the first few excited states.

In the argument of the exponential in (6.17), we expand

$$(6.18) \quad |\mathbf{X} + \frac{1}{2}\boldsymbol{\delta}| = X + \frac{1}{2}\hat{\mathbf{X}} \cdot \boldsymbol{\delta} + O(\delta^2/8X).$$

We then obtain, after some straightforward integrations,

$$(6.19) \quad V(\mathbf{X}, \mathbf{p}) \propto \frac{\exp[-(2X/\mu) - i(\mathbf{X} \cdot \mathbf{p}/2\hbar)]}{X^2} \cdot \left\{ \frac{1}{[1 - (1/2\beta)^2]} + O \left[\frac{3\mu(1 + (\beta/2)^2)}{8X(1 - (\beta/2)^2)} \right] \right\},$$

where

$$(6.20) \quad \beta^2 = \left(\hat{\mathbf{X}} + i \frac{3\mu}{4\hbar} \mathbf{p} \right)^2.$$

The momentum enters into (6.19) in the exponential and in β . In the exponential, putting $X = \mu = 1.16 \cdot 10^{-13}$ cm (a typical distance in the range in which the exchange interaction will be most important) and $|\mathbf{p}| = (8mK)^{\frac{1}{2}}$, with $K \cong 1.1$ MeV, we have $(\mathbf{X} \cdot \mathbf{p}/2\hbar) \cong 0.26$. Moreover, we have $\beta = 1$ for $\mathbf{p} = 0$, and β does not differ appreciably from unity while $(3\mu|\mathbf{p}|/4\hbar) \lesssim 1$. With the same substitution for $|\mathbf{p}|$ as before, we have $(3\mu|\mathbf{p}|/4\hbar) \cong 0.4$. The velocity dependence of the exchange interaction is therefore not very important in the ground state, but may well be more important in the excited states.

The two contributions to the velocity dependence are of different types. The term $(-i\mathbf{X} \cdot \mathbf{p}/2\hbar)$ in the exponential is of purely kinematic origin, and arose from the transformation to internal co-ordinates of the α -clusters. The terms containing β are related to the ratio $\omega_0/\omega_{\text{int}}$ introduced in I, section 4. This can be seen by rewriting $(3\mu|\mathbf{p}|/4\hbar)$ in the form

$$(6.21) \quad \left(\frac{3\mu|\mathbf{p}|}{4\hbar} \right) = \frac{3}{4\hbar} \left(\frac{2\hbar^2}{3m|E_0^{(1)} - E^{(b)}|} \right)^{\frac{1}{2}} \cdot \left(8m \cdot \frac{1}{4} \hbar \omega_0 \right)^{\frac{1}{2}} = \frac{\sqrt{3}}{2} \sqrt{\left(\frac{\omega_0}{\omega_{\text{int}}} \right)}.$$

Equation (6.21) is only valid in the ground state; the right hand side must be multiplied by $[(s + \frac{1}{2})/\frac{1}{2}]^{\frac{1}{2}}$ for the s -th excited state of relative vibration.

Putting $\beta = 1$ in (6.19), and $(\mu/X) \cong 1.2/3.7 = 0.3$, we find that the O term in the bracket $\{ \}$ in (6.19) is about one quarter of the first term.

Bearing in mind that the term (6.17) was the longest-range term in the exchange interaction, we adopt as a rough approximation for the asymptotic exchange interaction the formula

$$(6.22) \quad V_x(X) \cong \mathcal{G}_x \frac{\exp[-2X/\mu]}{X^2},$$

where \mathcal{G}_x is a constant. In using this procedure, we are assuming that if all the exchange terms were calculated and added together, there would be no chance cancellation between them so as to lead to a composite distance behaviour substantially different from that of a single term.

From (6.22) and the remarks at the beginning of this section, we see that the asymptotic ranges of the direct, polarisation, and exchange interactions are respectively (for a π -meson Yukawa potential) 1.4, 0.7 and $0.6 \cdot 10^{-13}$ cm. (Here we have considered only the exponential factor).

7. - Additivity of the Interactions.

We now discuss the additivity of the three types of interaction in turn. The direct interaction (3.27) can be generalised for an assembly of α -clusters. We obtain

$$(7.1) \quad \langle 0 | V_{\alpha\alpha}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) | 0 \rangle = \\ = \int \Psi_0^{(\alpha)*}(1) \dots \Psi_0^{(\alpha)*}(n) \left(\sum_{i>j} V_{\alpha i \alpha j} \right) \Psi_0^{(\alpha)}(1) \dots \Psi_0^{(\alpha)}(n) dv_{\text{int}}^{(1)} \dots dv_{\text{int}}^{(n)},$$

where $\mathbf{X}_1, \dots, \mathbf{X}_n$ are the mass centres of the clusters; $\Psi_0^{(\alpha)}(i)$ is the ground state internal wave function of the α -cluster whose mass centre is fixed at \mathbf{X}_i ; $V_{\alpha i \alpha j}$ is the interaction potential between the nucleons in clusters i and j , and $\int dv_{\text{int}}^{(i)}$ signifies integration over the internal co-ordinates of the i -th cluster.

The right hand side of (7.1) breaks up into a sum of contributions from pairs:

$$(7.2) \quad \langle 0 | V_{\alpha\alpha} | 0 \rangle = \sum_{i>j} \int \Psi_0^{(\alpha)*}(i) \Psi_0^{(\alpha)*}(j) V_{\alpha i \alpha j} \Psi_0^{(\alpha)}(i) \Psi_0^{(\alpha)}(j) dv_{\text{int}}^{(i)} dv_{\text{int}}^{(j)}.$$

Since each integral is to be calculated with the cluster mass centres fixed,

(7.2) is a static potential. The summation over pairs in (7.2) implies that $\langle 0 | V_{\alpha\alpha} | 0 \rangle$ is additive.

The exchange interaction is not additive. This is evident from a consideration of the analogous electron exchange interaction between molecules: two H atoms can form an H_2 molecule, but a third H atom cannot be added.

We now consider the additivity of the polarisation interaction. We saw in section 4 that the Green's function occurring in the polarisation integral for more than two α -clusters is a complicated function of all the bond distances. The velocity dependence of the polarisation interaction is therefore not additive. However, since the velocity dependent contribution to the ground state energy is probably not more than a fraction of 1 MeV, we neglect it in the rest of this section. We consider three α -clusters containing nucleons (1, 2, 3, 4), (5, 6, 7, 8), (9, 10, 11, 12), respectively. The polarisation operator in the generalised equation (4.13) then becomes

$$(7.3) \quad V_{\text{pol}}(\mathbf{X}) \cong - \sum_{\substack{i,j,k \\ \text{not all zero}}} \frac{\langle 000 | V_{\alpha\alpha} | ijk \rangle \langle ijk | V_{\alpha\alpha} | 000 \rangle}{E_{ijk} - E}.$$

Here $|ijk\rangle$ is the product of internal wave functions of the three clusters with the first, second and third α -clusters in states i, j, k respectively, and E_{ijk} stands for $(E_i^{(1)} + E_j^{(1)} + E_k^{(1)})$. We have written

$$(7.4) \quad V_{\alpha\alpha} = V_{\alpha_1\alpha_3} + V_{\alpha_2\alpha_3} + V_{\alpha_3\alpha_1}.$$

If we introduce (7.4) into (7.3), the polarisation term becomes

$$(7.5) \quad - \sum_{\mu > \nu} \sum_{\varrho > \sigma} \sum_{\substack{i,j,k \\ \text{not all zero}}} \frac{\langle 000 | V_{\alpha\mu\alpha\nu} | ijk \rangle \langle ijk | V_{\alpha\varrho\alpha\sigma} | 000 \rangle}{(E_{ijk} - E)}.$$

Using the orthogonality properties of the wave functions, we can write (7.5) in the form

$$(7.6) \quad - \sum_{\substack{i,j \\ \text{not both zero}}} \frac{\langle 000 | V_{\alpha_1\alpha_2} | ij0 \rangle \langle ij0 | V_{\alpha_1\alpha_2} | 000 \rangle}{(E_{ij0} - E)} + \text{similar terms (23) and (31)} \\ - \sum_{i \neq 0} \frac{\langle 000 | V_{\alpha_1\alpha_3} | i00 \rangle \langle i00 | V_{\alpha_1\alpha_3} | 000 \rangle}{(E_{i00} - E)} + \text{similar terms (21, 23) and (31, 32)}.$$

Since we have $2E \cong 3E_0^{(1)}$, the first set of terms in (7.6) is additive while the second set is not. The non-additive terms vanish if

$$(7.7) \quad \int \Psi_0^{(\alpha)*}(1) V_{\alpha_1\alpha_2} \Psi_0^{(\alpha)}(1) d\mathbf{r}_{\text{int}}^{(1)} = 0$$

for all pairs of α -clusters. Whether (7.7) is satisfied depends on the spin and charge dependence of the nuclear potential.

8. - A Note on the Practical Calculation of Forces.

The methods of sections 3 and 4 provide means of bracketing the expectation values of the polarisation and direct interactions by rough calculations. (See also ⁽²⁾, p. 174). For a given assumed analytic form of Φ , the lowest eigen value E'_0 of equation (3.30) always lies above the correct value E_0 . The variational approach therefore gives a lower limit for the numerical value of the sum of the direct and polarisation forces.

According to (5.1), the static approximation obtained from the perturbation approach gives an upper limit for the numerical value of the expectation-value of the polarisation interaction if the velocity dependent effects are not too large. If we put

$$(1/|E - E_i^{(1)} - E_j^{(1)}|) \delta(|\mathbf{X} - \boldsymbol{\xi}|) \quad \text{for} \quad (\mu/2\pi\hbar^2) \exp [(-k_{ij}|\mathbf{X} - \boldsymbol{\xi}|)/|\mathbf{X} - \boldsymbol{\xi}|]$$

in (4.13), we find that

$$\begin{aligned} & \left| \text{Expectation value of} \right. \\ & \left. \left[\frac{\mu}{2\pi\hbar^2} \sum' \langle 00 | V_{\alpha\alpha}(\mathbf{X}) | ij \rangle \frac{\exp [-k_{ij}|\mathbf{X} - \boldsymbol{\xi}|]}{|\mathbf{X} - \boldsymbol{\xi}|} \langle ij | V_{\alpha\alpha}(\boldsymbol{\xi}) | 00 \rangle \right] \right| \\ & \leq \left| \text{Expectation value of} \right. \\ & \left. \left[\sum' \frac{|\langle 00 | V_{\alpha\alpha}(\mathbf{X}) | ij \rangle|^2}{|E_i^{(1)} + E_j^{(1)} - E|} \right] \right| \\ & \leq \left| \text{Expectation value of} \right. \\ (8.1) \quad & \left. \frac{1}{\Delta E} [\langle 00 | V_{\alpha\alpha}^2(\mathbf{X}) | 00 \rangle - |\langle 00 | V_{\alpha\alpha}(\mathbf{X}) | 00 \rangle|^2] \right|, \end{aligned}$$

where

$$(8.2) \quad \Delta E = |E_i^{(1)} + E_j^{(1)} - E|_{\min},$$

which is the minimum value of this quantity occurring in the sum \sum' . In the last step of (8.1) we have replaced all denominators by ΔE and used the completeness relation. The perturbation approach therefore leads to an upper limit for the numerical value of the polarisation forces. If the direct inter-

action is attractive, or repulsive and smaller in magnitude than the polarisation interaction, then the perturbation method gives an upper limit to the numerical value of the sum of the two.

APPENDIX

Investigations of the α -Clustering Process.

The hypothesis that the wave function (2.1) is a good approximation has been examined by several authors. ROSENFELD ⁽⁷⁾ (Ch. XII) discussed the Fermi gas model in which the wave function is built up from individual particle functions whose spatial parts are plane waves. The zero order wavefunction, when antisymmetrized in accordance with the exclusion principle, shows that nucleons of the same spin and charge tend to be separated from one another. In the next approximation, which takes account of the nuclear interaction, one finds that with the nucleon-nucleon potential of the form

$$(A.1) \quad V_{\alpha ij} = \mathcal{G} V(r_{ij}) \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j (a_{\tau} + a_{\sigma\tau} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j),$$

the tendency for nucleons of the same spin and charge to be separated is increased, while all other nucleon pairs are drawn together. This sort of positional correlation of nucleons according to their spin and charges is just what is implied by (2.1).

WHEELER ⁽⁸⁾ considered the relation between the single particle and α -particle models in the nucleus ^{16}O . He showed that if the wave function Ψ is written as a Slater determinant built up from single particle states (one s and three p states), Ψ can be re-expressed, without changing its value, in a form representing four α -clusters at the apices of a tetrahedron. This is another aspect of the positional correlation shown by the Fermi gas model in zero order. The effect of the nuclear interactions may then be expected to reinforce this clustering, and to lead to the tetrahedral structure for ^{16}O suggested by the α -cluster model.

A calculation by GRÖNBLOM and MARSHAK ⁽⁹⁾ for the nucleus ^{12}C started from the α -cluster model. They set up a zero order wave function in which the twelve nucleons are grouped into three clusters at the apices of a fixed equilateral triangle. They introduced, as a perturbation, the difference between a potential which will lead to such a configuration and an assumed nucleon-nucleon potential. They found that the next approximation to the wave function represents a large degree of « solution » of the clusters; for example, a proton from one cluster and a neutron from another go over into an excited state of the third. The second order correction to the energy in this perturbation procedure turns out to be of the order of magnitude of the first order energy. This result was considered by the authors to provide an argument against the assumption of a wave function of the type (2.1) and against the possibility of any simple explanation of the apparent additivity of α -cluster interactions discussed in I. Actually, the result is to be expected from the way in which the calculation was set up. The triangular configuration which the α -cluster model suggests must, in the O^+ ground state of ^{12}C , be orientated

at random in space. If this wave function were expanded in terms of a system of functions based on the fixed triangle, the result would be very complicated, and could not be expected to show any simple features such as clustering about specific fixed points. Hence, even if the α -cluster model is the consequence of the nucleon-nucleon interaction assumed by GRÖNBLÖM and MARSHAK, this result could not be expected to show up in any simple way in the system of basic functions they used.

WERGELAND (*) attacked the problem of lability of the two α -clusters in ${}^8\text{Be}$. He considered that the wave function for this nucleus must be somewhere between that appropriate to an independent particle model and that corresponding to a grouping of the eight nucleons into two α -clusters. He therefore set up a wave function of the form:

$$(A.1) \quad \Psi = \Psi^{(s)} + \gamma \Psi^{(2\alpha)}.$$

Here $\Psi^{(s)}$ is a properly antisymmetrized and normalized function containing four nucleons in the ground and four in the first excited state of a simple harmonic oscillator. $\Psi^{(2\alpha)}$ is a normalized wave function of the type (2.1) with the two clusters centred about two fixed points. γ is treated as a variational parameter. Using a pure Gaussian Majorana potential, WERGELAND first adjusted the parameters to minimise the energies corresponding to the two parts of (A.1) taken separately. He then found a repulsion of the α -clusters with the complete wave function. However, on adjusting the parameters in $\Psi^{(2\alpha)}$ to make $E^{(s)}$ equal to $2E_0^{(1)}$, he found that the system is stable, but contains about 45 percent of the wave function orthogonal to $\Psi^{(2\alpha)}$ i.e.,

$$(A.2) \quad 1 - \frac{|\langle \Psi^{(2\alpha)} | \Psi \rangle|^2}{|\langle \Psi | \Psi \rangle|^2} \cong 0.45.$$

This result again is to be expected from the assumptions made. In adjusting the energy of an α -particle in the ground state to be equal to half $E^{(s)}$, the high binding energy, which would be the main factor in bringing about strong clustering, is thrown away. Moreover, the centering of the α -clusters on two fixed points assigns an unrealistically large kinetic energy to the function $\Psi^{(2\alpha)}$; this effect again leads to a reduction of the contribution of $\Psi^{(2\alpha)}$ to the wave function. (The ${}^8\text{Be}$ nucleus should be pictured as a dumbbell whose axis is orientated at random).

RIASSUNTO (*)

Si sviluppa per l'interazione delle particelle α nei nuclei un formalismo analogo a quello adottato per l'interazione degli atomi. Le forze reali possono separarsi in un'interazione diretta (analogo all'interazione coulombiana degli ioni a grande distanza), un'interazione di scambio fra nucleoni (analogo allo scambio fra elettroni), e un'inte-

(*) Traduzione a cura della Redazione.

razione di polarizzazione (analoga alle forze di van der Waals). L'interazione diretta è additiva e indipendente dalla velocità. L'interazione di scambio fra nucleoni dipende dalla velocità e non è additiva. L'interazione di polarizzazione dipende dalla velocità e la sua additività è determinata dalla dipendenza dello spin e della carica dal potenziale nucleare. Si descrive il passaggio dal formalismo descritto al trattamento consueto delle forze atomiche di van der Waals. Non è probabile la presenza di effetti di ritardo dovuti alla natura ondulatoria del campo delle forze nucleari. Si dimostra che la forma asintotica delle forze d'interazione tra due particelle α e il potenziale di un mesone π di Yukawa contiene fattori esponenziali $\exp[-s/1,4]$ (diretta), $\exp[-s/0,7]$ (polarizzazione), $\exp[-s/0,6]$ (scambio), dove s è l'intervallo fra i centri di massa in unità 10^{-13} cm. Si portano argomenti contro le conclusioni di precedenti lavori che asserivano che gli aggregati di α nella materia nucleare debbano dissolversi.

Sul metodo degli elettroni d'urto per la determinazione della massa di particelle al minimo di ionizzazione.

A. DE MARCO, A. MILONE e M. REINHARZ

Istituto di Fisica dell'Università - Genova

Istituto Nazionale di Fisica Nucleare - Sezione Aggregata di Genova

(ricevuto il 7 Marzo 1955)

Riassunto. — È stata studiata la possibilità dell'applicazione, nelle emulsioni nucleari, di un metodo per la determinazione della massa di particelle al minimo di ionizzazione, nel caso che queste diano origine ad elettroni d'urto. L'applicazione dei teoremi di conservazione dell'energia e dell'impulso, permettono infatti di determinare una grandezza, funzione della velocità della particella, che varia sensibilmente al variare della velocità stessa. Sono stati fissati dei criteri di carattere generale per il riconoscimento degli elettroni d'urto, sono state esaminate le diverse cause di errore che influenzano la misura sia dell'energia che dell'angolo di emissione di questi elettroni e calcolato il corrispondente errore sulla massa della particella urtante; l'analisi della dispersione dei dati ottenuti sperimentalmente dà risultati in buon accordo con il valore dell'errore calcolato secondo i criteri da noi suggeriti. Tenendo conto della frequenza degli eventi nei quali si genera un elettrone d'urto riconoscibile secondo i nostri criteri, sono state discusse le limitazioni del metodo e le sue possibili applicazioni sia nel caso di un singolo evento, sia nell'impiego sistematico per la determinazione della composizione isotopica dei nuclidi che fanno parte della radiazione cosmica primaria.

1. — Introduzione.

La recente scoperta che la radiazione cosmica primaria contiene un'importante percentuale di nuclei di vario numero atomico ha reso importante la ricerca di metodi che consentano di studiare, da questo punto di vista, la composizione di questa radiazione. Nella quasi totalità dei lavori finora eseguiti in questo campo i nuclei della radiazione primaria sono stati distinti solo per

la loro carica, mentre sarebbe di grande interesse la possibilità di misurare, con buona approssimazione, anche la loro massa. Tale possibilità esiste notoriamente per particelle di bassa velocità, ma lo studio di queste non è interessante per lo scopo che ci proponiamo, perchè una grossa aliquota dei nuclei di bassa velocità ha certamente origine secondaria.

I metodi usati normalmente per la determinazione della massa, in emulsioni nucleari, si basano tutti sulla misura di due tra le seguenti tre grandezze ^(1,2): il range residuo R , la ionizzazione specifica I/I_0 (o meglio la densità specifica dei grani N/N_0) e l'angolo medio di scattering $\bar{\alpha}$, dalle quali si può risalire rispettivamente al valore della energia cinetica, della velocità e del prodotto del momento per la velocità ($p\beta$). Queste grandezze sono però sufficienti a determinare la massa solo se la particella è alla fine del suo percorso o se la sua velocità è abbastanza piccola ($\gamma = E_t/mc^2 < 2$); per velocità più elevate ove la ionizzazione specifica non varia apprezzabilmente al variare della velocità, si potrebbe ricorrere invece alla misura della curvatura della traiettoria della particella in un campo magnetico ⁽³⁻¹⁰⁾, dalla quale si risale direttamente al valore del momento p . Questo metodo però si è mostrato a tutt'oggi difficilmente utilizzabile perchè con i campi magnetici attualmente disponibili, la curvatura della traiettoria risulta fortemente disturbata dallo scattering e dalle distorsioni, in modo che nella maggioranza dei casi è solo possibile determinare il segno della carica della particella in esame.

Un metodo che permette di determinare una grandezza che è funzione della velocità, anche quando la particella è al minimo di ionizzazione, è quello di utilizzare le informazioni date dagli urti elastici che la particella subisce con gli elettroni della materia che essa attraversa. Questo metodo è stato usato da LEPRINCE-RINGUET e coll. ^(11,12), per determinare la massa di alcuni mesoni dei raggi cosmici fotografati in camera di Wilson, con campo magnetico.

⁽¹⁾ G. P. S. OCCHIALINI: *Nuovo Cimento*, **6**, 413 (1949).

⁽²⁾ C. F. POWELL: *Reports on Progress in Physics*, **13**, 350 (1950).

⁽³⁾ C. F. POWELL e S. ROSENBLUM: *Nature*, **161**, 473 (1948).

⁽⁴⁾ I. BARBOUR: *Phys. Rev.*, **76**, 320 (1949).

⁽⁵⁾ C. FRANZINETTI: *Phil. Mag.*, **41**, 86 (1950).

⁽⁶⁾ Y. GOLDSCHMIDT-CLERMONT e M. MERLIN: *Nuovo Cimento*, **7**, 220 (1950).

⁽⁷⁾ C. DILWORTH, S. I. GOLDSACK, Y. GOLDSCHMIDT-CLERMONT e F. LEVI: *Phil. Mag.*, **41**, 1032 (1950).

⁽⁸⁾ M. MERLIN, B. VITALE e I. GOLDSCHMIDT-CLERMONT: *Nuovo Cimento*, **9**, 421 (1952).

⁽⁹⁾ C. DILWORTH e S. J. GOLDSACK: *Nuovo Cimento*, **10**, 926 (1953).

⁽¹⁰⁾ G. SOMEDA e M. MERLIN: *Nuovo Cimento*, **9**, 73 (1954).

⁽¹¹⁾ L. LEPRINCE-RINGUET, S. GORODETZKI, E. NAGEOTTE e R. RICHARD-FOY: *Phys. Rev.*, **59**, 460 (1941); *Journ. de Phys. et Rad.*, **2**, 63 (1942).

⁽¹²⁾ L. LEPRINCE-RINGUET e M. LHÉRITIER: *Compt. Rend.*, **219**, 618 (1944); *Journ. Phys. et Rad.*, **7**, 65 (1946).

Disgraziatamente questi urti sono troppo rari perchè il metodo possa essere usato su larga scala, soprattutto in camera di Wilson, dove la quantità di materia che la particella attraversa si riduce nella maggioranza dei casi a pochi centesimi di g/cm^2 . Inoltre solo una piccola percentuale di tali eventi si presenta in maniera da essere utilizzabile per uno studio accurato.

Abbiamo pensato che questo metodo possa avere una maggiore utilità se applicato alle particelle rivelate con le emulsioni fotografiche, perchè l'uso delle emulsioni cosiddette « stripped » consente di seguire per parecchi centimetri una traccia che si sviluppa in un mezzo di densità abbastanza elevata. È facile calcolare che in tali emulsioni la probabilità di rivelare un urto elastico è almeno cento volte più grande che nella camera di Wilson.

Il metodo da noi studiato consiste nel risalire, attraverso la misura dell'angolo di emissione e della energia cinetica dell'elettrone urtato, ad una grandezza che è funzione della velocità e della massa della particella urtante; il valore di questa grandezza unitamente a quello del $p\beta$ della particella urtante, ottenuto attraverso la misura dell'angolo medio di scattering, permette di determinare il valore della sua massa.

2. - Considerazioni generali.

Se con m_e e con m si indicano rispettivamente la massa dell'elettrone urtato e quella della particella urtante, espresse in MeV/c^2 , con E_e l'energia in MeV dell'elettrone urtato, con ω il suo angolo di emissione e con p il momento della particella primaria in MeV/c , si ha dai teoremi di conservazione dell'energia e dell'impulso ⁽¹³⁾, ponendo $c = 1$:

$$(1) \quad \frac{p^2}{m_e^2 + m^2 + 2m_e(p^2 + m^2)^{\frac{1}{2}}} = \frac{1}{U},$$

ove con U si è indicata la quantità

$$(2) \quad U = \left(1 + \frac{2m_e}{E_e}\right) \cos^2 \omega - 1,$$

che dipende solo dalle grandezze relative all'elettrone urtato. La (2) può essere espressa in funzione dell'angolo medio di scattering per 100μ , $\bar{\alpha}_e$, che è la quantità direttamente misurata: in tal caso, se K è la costante di scattering si ha:

$$(2') \quad U = [g + (1 + g)^{\frac{1}{2}}] \cos^2 \omega - 1,$$

⁽¹³⁾ B. ROSSI: *High Energy Particles* (New York, 1952), pag. 14.

avendo posto

$$g = \bar{\alpha}_e / K.$$

Se la particella urtante è un elettrone, il primo membro della (1) dà un numero eguale alla sua energia cinetica espressa in MeV; se invece $m \gg m_e$, associando la (1) alla $\bar{\alpha}_p = KZ/p\beta$, ove $\bar{\alpha}_p$ è l'angolo medio di scattering per 100μ della particella primaria di carica Z , si ha:

$$(3) \quad m = \frac{1}{\sqrt{2}} [A + (A^2 - B)^{\frac{1}{2}}]^{\frac{1}{2}},$$

ove A e B sono delle quantità funzioni delle grandezze direttamente misurabili:

$$A = \left(\frac{U}{G} - 1 \right)^2 + \frac{1}{G} \left(\frac{U}{G} - 1 \right) - \frac{1}{2}, \quad B = \frac{1}{4} - \frac{1}{G} \left(\frac{U}{G} - 1 \right),$$

nelle quali G indica la quantità

$$1/p\beta = \bar{\alpha}_p / KZ.$$

La (3) si semplifica notevolmente tutte le volte in cui è soddisfatta la condizione

$$(4) \quad \frac{m^2 - m_e^2}{2m_e} \gg E,$$

dove E è l'energia totale della particella urtante; in tal caso il primo membro della (1) dà il valore di $(p/m)^2 = \beta^2/(1 - \beta^2)$, cioè una grandezza funzione della sola velocità della particella urtante, che per β vicini ad 1, varia fortemente per piccole variazioni di β . L'errore che si commette nel calcolo di p/m , quando si accetta questa approssimazione, è dato da

$$1 - \left(1 + 2 \frac{m_e E}{m^2} \right)^{-\frac{1}{2}},$$

e questo errore si mantiene piccolo nella maggioranza dei casi che interessano: per esempio, l'errore risulta del 10 % per mesoni leggeri con energia $E \ll 22$ GeV o per protoni con energia $E \ll 10^3$ GeV.

Se è soddisfatta la condizione (4), la (3) diventa:

$$(5) \quad m = \frac{[U(U+1)]^{\frac{1}{2}}}{\bar{\alpha}_p} KZ,$$

che dà la massa della particella urtante in funzione delle quantità misurabili $\bar{\alpha}_p$, $\bar{\alpha}_e$, ω .

3. — Considerazioni sperimentali.

L'applicazione di questo metodo richiede: 1) di poter riconoscere gli elettroni d'urto e distinguerli dagli eventi spuri dovuti agli elettroni di bassa energia che casualmente hanno origine nelle immediate vicinanze della particella in esame; 2) di poter determinare l'angolo di emissione di questi elettroni e 3) di poter determinare la loro energia cinetica E_e . Queste operazioni contengono un certo numero di elementi di arbitrarietà che è necessario ridurre per avere la minima influenza sui risultati; pertanto ciascuna di esse deve essere discussa in dettaglio.

3.1. *Criteri per il riconoscimento degli elettroni d'urto.* — Dato il grande numero di elettroni di fondo che si trovano nelle immediate vicinanze della traccia in esame, occorre innanzi tutto stabilire dei criteri che evitino, per quanto è possibile, di prendere come traccia di un elettrone d'urto un segmento di traccia di un elettrone di fondo. In considerazione di ciò abbiamo deciso di richiedere che una traccia per essere attribuibile ad un elettrone d'urto: *a)* sia emessa nella direzione di moto della particella primaria e formi quindi con essa un angolo acuto; *b)* abbia inizio nelle immediate vicinanze della traccia della particella primaria. Per precisare quest'ultima richiesta abbiamo stabilito di ritenere che un grano sia il primo di una traccia che ivi si inizia, quando entro un raggio di 150 μ da questo non ve ne siano altri che possano appartenere ad una traccia che verosimilmente si prosegue nella data. La distanza di 150 μ è stata scelta sia perchè, con l'ingrandimento adoperato nella ricerca di questi eventi, essa rappresenta circa il diametro di un campo, sia perchè è estremamente improbabile trovare lacune così grandi in emulsioni normalmente sviluppate e con tracce di particelle che non abbiano subito effetti di fading. Una particella sarà pertanto da noi considerata un elettrone d'urto, quando la sua traccia è complanare con quella della particella urtante, nei limiti concessi dalla profondità focale, e non inizia ad una distanza superiore a 15 μ da quest'ultima.

Per valutare poi la frequenza di eventuali associazioni casuali abbiamo determinato sperimentalmente la probabilità di incontrare un elettrone isolato che abbia origine nella emulsione, esaminando (*) una lunghezza di 50 cm di traccia fittizia, nella quale la direzione di moto della particella urtante era data dalla linea verticale del reticolo ed imponendo agli elettroni di obbedire alle condizioni *a)* e *b)*. In detta lunghezza sono stati trovati 4 elettroni soddisfacenti a queste condizioni; è stato però notato che la loro energia era minore o tutt'al più eguale ad un MeV.

(*) La ricerca è stata eseguita sulle stesse emulsioni ove poi sono stati cercati gli elettroni d'urto.

Si è quindi deciso di considerare utilizzabile ai fini della misura solo elettroni con un'energia maggiore di 4 MeV. La scelta di questo taglio è stata fatta anche in base alla constatazione della grande difficoltà che si ha nel seguire elettroni di energia più bassa. Inoltre, solo per energie maggiori di questa, le misure dell'angolo medio di scattering effettuate con il normale metodo angolare sono sufficientemente riproducibili.

3.2. *Criteri per la misura dell'angolo di emissione.* - Un'altra difficoltà incontrata è stata quella di determinare l'angolo di emissione ω dell'elettrone urtato e di dare una valutazione dell'errore da attribuire alla misura di quest'angolo. Qui la difficoltà proviene essenzialmente dal fatto che gli elettroni d'urto hanno, nella maggioranza energie piuttosto basse e quindi angoli medi di scattering abbastanza notevoli (7° per 100μ per energie di 4 MeV). Ciò disturba notevolmente la misura dell'angolo di emissione e siamo stati quindi condotti ad esaminare le cause principali di errore, che intervengono nella misura dell'angolo formato da due tracce, in emulsione nucleare, allo scopo di ricercare le migliori condizioni sperimentali in cui eseguire la misura stessa. L'angolo ω , quando si adoperino tracce di primario sufficientemente lunghe e quindi poco inclinate sul piano della lastra, si può ricavare con buona approssimazione dalla relazione $\cos \omega = \cos \theta \cos \varphi$. Gli angoli θ e φ sono le proiezioni di ω rispettivamente su i piani parallelo e perpendicolare al piano della emulsione, al quale con buona approssimazione può ritenersi appartenga la particella urtante.

La misura dell'angolo θ si ottiene per differenza di due letture θ_1 e θ_2 , fatte allineando il reticolo dell'oculare goniometrico successivamente sulle due tracce. Le sorgenti principali di errore, nella misura di ognuno di questi angoli sono essenzialmente;

1) la dimensione finita dei grani e la loro distribuzione intorno alla traiettoria con la quale si allinea il reticolo. Questo errore, come è noto ⁽¹⁴⁾, è dato da $\varepsilon_1 = kl^{-\frac{3}{2}}$, ove k è una costante che dipende dal diametro dei grani e dalla loro densità lineare, ed l la lunghezza di traccia che serve per l'allineamento del reticolo;

2) lo scattering della particella, nella lunghezza l , che dà luogo ad un errore ⁽¹⁵⁾ $\varepsilon_2 = \bar{\alpha}(l/300)^{\frac{1}{2}}$, ove $\bar{\alpha}$ indica l'angolo medio di scattering per 100μ , espresso in radianti;

3) l'errore di lettura del goniometro che nella maggioranza dei casi è trascurabile rispetto ai precedenti.

⁽¹⁴⁾ Y. GOLDSCHMIDT-CLERMONT: *Nuovo Cimento*, **7**, 3-bis, 331 (1950); R. LEVI SETTI: *Nuovo Cimento*, **8**, 96 (1951).

⁽¹⁵⁾ B. ROSSI e K. GREISEN: *Rev. of Mod. Phys.*, **13**, 268 (1941).

Essendo questi errori indipendenti tra loro si può prendere come stima dell'errore totale, per ognuno dei due allineamenti, l'espressione

$$(6) \quad \sqrt{\frac{k^2}{l^3} + \frac{\bar{\alpha}^2}{300}} l,$$

e quindi l'errore totale su θ sarà:

$$\Delta\theta = \sqrt{\Delta\theta_1^2 + \Delta\theta_2^2},$$

Il valore dell'angolo φ si ricava per differenza dai valori delle inclinazioni ψ_1 e ψ_2 delle due tracce sul piano della lastra e tali inclinazioni si ricavano a loro volta dalla $\psi = \text{arctg } \Delta q/L = \text{arctg } \varrho$, dove Δq rappresenta la differenza di quota di due punti di una traccia ed L la proiezione della loro distanza sul piano dell'emulsione.

Gli errori di misura in queste determinazioni sono essenzialmente dovuti:

- 1) alla messa a fuoco di un grano della traccia ed alla lettura sul nonio del microscopio ($\varepsilon_1 = \Delta z$);
- 2) alla imprecisione nella misura di L ($\varepsilon_2 = \Delta L$);
- 3) allo scattering della traccia in detta lunghezza ($\varepsilon_3 = (\bar{\alpha}/\sqrt{300})L^{\frac{1}{2}}$);
- 4) alla non uniformità dello spessore S dell'emulsione prima dello sviluppo ($\varepsilon_4 = \Delta S$; questo errore è circa del 5%);
- 5) alla non costanza del passo della vite che governa la messa a fuoco del microscopio;
- 6) ai cambiamenti dell'umidità dell'aria che deformano in prevalenza i primi strati dell'emulsione.

Non tenendo conto degli errori 5) e 6), che richiederebbero un'analisi a parte e che comunque possono essere ridotti di piccola entità, l'errore su ciascuno dei due valori di ϱ è dato da:

$$(7) \quad \sqrt{\varrho^2 \left[\left(\frac{\Delta S}{S} \right)^2 + \left(\frac{\Delta L}{L} \right)^2 \right] + 2 \left(\frac{\Delta z}{z} \right)^2 + \frac{\bar{\alpha}^2}{300}} L,$$

e quindi l'errore totale è: $\Delta\varrho = (\Delta\varrho_1^2 + \Delta\varrho_2^2)^{\frac{1}{2}}$.

Le condizioni ottime per la determinazione degli angoli θ e φ si ricavano dalle (6) e (7) cercando per quali valori di l ed L esse hanno un minimo. Detti valori sono:

$$(6') \quad l = 5,5 \left(\frac{k}{\bar{\alpha}} \right)^{\frac{1}{3}},$$

$$(7') \quad L = 8,43 \left(\frac{\varrho(\Delta L)^2 + 2(\Delta z)^2}{\bar{\alpha}^2} \right)^{\frac{1}{2}},$$

ove ϱ è un valore approssimato ottenuto con una lunghezza di cella L qualsiasi.

Gli errori su θ e su φ diventano allora

$$(8) \quad \Delta\theta_{\min} = \sqrt{2,43 \cdot 10^{-2} k^{\frac{1}{2}} (\bar{\alpha}_1^{\frac{3}{2}} + \bar{\alpha}_2^{\frac{3}{2}})},$$

$$(9) \quad \Delta\varphi_{\min} = \sqrt{\left(\frac{\Delta\varrho_{1\min}}{1 + \varrho_1} \right)^2 + \left(\frac{\Delta\varrho_{2\min}}{1 + \varrho_2} \right)^2},$$

dove

$$\Delta\varrho_{1,2\min} = \sqrt{\varrho_{1,2}^2 \left(\frac{\Delta S}{S} \right)^2 + 4,2 \cdot 10^{-2} (\varrho_{1,2}^2 \Delta L^2 + 2\Delta z^2)^{\frac{1}{2}} \bar{\alpha}_{1,2}^{\frac{4}{3}}}.$$

La (8) e la (9) danno, con buona approssimazione, i valori degli errori quadratici medi di cui sono affette le misure di θ e di φ ; il grado di approssimazione è essenzialmente stabilito da una giusta scelta delle costanti che in esse figurano e dalla precisione con cui è stato misurato l'angolo medio di scattering $\bar{\alpha}$. Prendendo per k il valore 0,5 (*), $\Delta L = 2\mu$ (†) per ogni spostamento del reticolo, $\Delta z = 1\mu$ e considerando utile al nostro scopo solo elettroni in cui è possibile misurare $\bar{\alpha}_e$ con un errore non superiore al 25%, le (8) e (9) permettono di valutare gli errori $\Delta\theta$ e $\Delta\varphi$ con una incertezza minore del 30%.

A questi errori, nel nostro caso, andrebbe aggiunto il contributo dovuto allo scattering dell'elettrone da parte dell'atomo da cui esso stesso proviene, ma come facilmente si può vedere (v. Appendice I) detto contributo è trascurabile.

Considerando ora il fatto che le varie cause di errore intervengono indipendentemente nella misura di θ e di φ (lo scattering nel piano orizzontale della emulsione è indipendente da quello nel piano verticale) è lecito, almeno nella maggioranza dei casi, calcolare l'errore su ω con la formula di Gauss: si ottiene

$$(10) \quad \Delta\omega = \cotg \omega (\tg^2 \theta \Delta\theta^2 + \tg^2 \varphi \Delta\varphi^2)^{\frac{1}{2}}.$$

La (10), risultando funzione di $\bar{\alpha}_e$, tiene conto anche dell'errore dovuto ai possibili scattering singoli che l'elettrone può subire appena emesso o nelle immediate vicinanze del punto di emissione.

(*) Il valore $k=0,5$ è stato ottenuto sperimentalmente, sulle stesse emulsioni dove poi sono stati cercati gli elettroni d'urto.

(†) Il valore $\Delta L=2\mu$ si potrebbe ridurre notevolmente, data l'accuratezza con cui si possono fare gli spostamenti del reticolo e la taratura di esso. In ogni caso però questo valore influenza poco l'errore. Al contrario Δz influisce sensibilmente ed il valore di 1μ è stato ottenuto mediante misure ripetute del dip.

3.3. *Considerazioni sulla determinazione dell'energia E_e .* — La determinazione dell'energia dell'elettrone urtato ottenuta attraverso la misura di $\bar{\alpha}_e$, è affetta da un errore sistematico dovuto al fatto che nella lunghezza di traccia su cui si effettua la misura, l'elettrone perde energia sia per ionizzazione che per radiazione. Le perdite di energia per ionizzazione conducono ad una sotto-stima del valore dell'energia. Questo errore, del quale in ogni caso si può tener conto non supera, per energie maggiori di 4 MeV il 15%.

Le perdite di energia per radiazione, come è noto, avvengono di preferenza in pochi atti elementari, in ciascuno dei quali, l'elettrone perde una parte considerevole della sua energia.

Gli elettroni, che hanno perduto energia per radiazione, sono quasi sempre riconoscibili a causa della brusca variazione del loro angolo medio di scattering; qualche volta invece non lo saranno o perchè la perdita di energia non è sufficientemente grande o perchè tale perdita avviene nel primo tratto della traccia. Un calcolo approssimativo della percentuale di elettroni che subiscono, a causa di urti radiativi, una perdita di energia frazionaria maggiore dell'errore percentuale statistico e di cui non ci si accorge dalle misure, mostra che tale percentuale è certamente minore del 5% (v. Appendice II). Questo fatto rappresenta, a nostro avviso, la limitazione più grave del metodo quando esso si voglia applicare al caso di un solo elettrone d'urto.

3.4. — In definitiva l'errore su U e sulla massa sono dati rispettivamente da:

$$(11) \quad \frac{\Delta U}{U} = \frac{U+1}{U} \sqrt{g^2 \left(\frac{\Delta \bar{\alpha}_e}{\bar{\alpha}_e} \right)^2 + 4 \operatorname{tg}^2 \omega \Delta \omega^2},$$

$$(12) \quad \frac{\Delta m}{m} = \sqrt{\left(\frac{1}{2} \frac{\Delta U}{U} \right)^2 + \left(\frac{\Delta \bar{\alpha}_p}{\bar{\alpha}_p} \right)^2}.$$

Come si vede dalla (11) l'errore percentuale su U aumenta rapidamente al crescere di ω ; conviene quindi limitarsi a considerare solo quegli elettroni emessi sotto un angolo sufficientemente piccolo ($< 20^\circ$) in modo che il contributo all'errore dato dalla misura dell'angolo ω non sia troppo grande. Essendo $U < 1$ la (11) ci mostra ancora che l'errore percentuale su U cresce al diminuire di U e quindi questo metodo sarà utilizzabile solo per particelle aventi un valore di U non troppo piccolo, cioè per particelle non troppo veloci.

4. — Risultati sperimentali.

4.1. — Abbiamo voluto controllare se l'entità degli errori calcolati nel paragrafo precedente era in accordo con una valutazione degli errori fatta direttamente analizzando i dati sperimentali. Si è dovuto rinunciare ad utilizzare

particelle di carica unitaria, che per la grande differenza percentuale delle masse da esse possedute si presenterebbero come le più adatte a questa verifica, perchè il numero di elettroni d'urto da esse prodotto risulta molto piccolo. Abbiamo perciò preferito studiare il caso di particelle con una carica molto elevata, per controllare se la distribuzione dei valori di U misurati sugli elettroni d'urto da esse prodotti era in accordo con la valutazione dell'errore quadratico medio da noi data nella (11). Si è rinunciato ad una verifica sperimentale sulla distribuzione dei valori di m perchè le condizioni sperimentali non si presentavano adatte; del resto, quest'ultimo controllo non avrebbe avuto alcun significato nei riguardi degli scopi del presente lavoro, perchè la validità della (12) è garantita dalla validità della (11).

La ricerca è stata eseguita in emulsioni « stripped » 600 μ esposte a 24500 m (2ª esposizione in Sardegna) su tre tracce che indicheremo con A , B , C , aventi diverse energie (la variazione di energia di ciascuna nel tratto considerato era trascurabile). Inizialmente sono stati presi in considerazione solo elettroni con un angolo $\theta \leq 20^\circ$, successivamente per ogni traccia, è stato stabilito un taglio per l'energia degli elettroni, come indicato nelle Tabelle. Nella foto 1 è dato qualche esempio di elettrone d'urto.

Per ogni elettrone sono state fatte, da due osservatori diversi, le misure di $\bar{\alpha}_e$, θ , φ ; il valore di $\bar{\alpha}_e$ è stato determinato col metodo angolare, apportando la correzione del rumore di fondo ed utilizzando, nella maggioranza dei casi, tutta la lunghezza della traccia, interrompendo le misure solamente ogni volta che si avevano dubbi sul suo proseguimento. L'errore è stato calcolato con la relazione

$$\frac{\Delta \bar{\alpha}}{\bar{\alpha}} = 0,05 + \left[\left(\frac{0,75}{\sqrt{N}} \right)^2 + \left(\frac{0,85}{2} \right)^2 \left(\frac{\varepsilon^2}{\bar{\alpha}_{sp}} \right)^2 \right]^{\frac{1}{2}},$$

essendo N il numero di misure indipendenti. La distorsione era trascurabile.

La misura dell'angolo φ è stata fatta con la lunghezza di cella data dalla (7'), mentre per la misura dell'angolo θ si è adottata la più piccola lunghezza di cella che permetteva un buon allineamento del reticolo, in quanto la (6') dava dei valori molto piccoli.

I risultati delle misure, relative alle tre tracce, sono date dalle tabelle I, II, III, nelle quali sono riportati, per ogni elettrone, rispettivamente l'angolo medio di scattering $\bar{\alpha}_e$; l'angolo ω ed il valore di U . Gli errori $\Delta \omega$ e ΔU sono stati calcolati con la (10) e la (11).

4.2. - Dall'esame di dette tabelle si vede subito che gli errori calcolati rendono conto abbastanza bene della dispersione dei vari valori di U trovati. Una conferma più quantitativa la si può avere confrontando tra loro l'errore di cui è affetta la media ponderata dei vari valori di U , ottenuta sia attraverso la consistenza interna (cioè direttamente dagli errori calcolati) sia attra-



TAV. I.

TABELLA I.

(Traccia A: lunghezza 3,1 cm; inclinazione $0^{\circ}37'$; $Z \simeq 16$; $\bar{\alpha}_e < (1,5)^{\circ}$).

| n | $\bar{\alpha}_e$ | $\omega \pm \Delta\omega$ | $(U \pm \Delta U) \cdot 10^4$ |
|-----|------------------|------------------------------------|-------------------------------|
| 1 | $1,20 \pm 0,20$ | $10^{\circ} 15' \pm 0^{\circ} 49'$ | 53 ± 81 |
| 2 | $1,33 \pm 0,20$ | $11^{\circ} 16' \pm 0^{\circ} 42'$ | 27 ± 79 |
| 3 | $0,79 \pm 0,13$ | $7^{\circ} 15' \pm 0^{\circ} 19'$ | 87 ± 46 |
| 4 | $0,16 \pm 0,21$ | $8^{\circ} 13' \pm 0^{\circ} 31'$ | 155 ± 71 |
| 5 | $0,50 \pm 0,07$ | $6^{\circ} 20' \pm 0^{\circ} 20'$ | 53 ± 25 |
| 6 | $0,82 \pm 0,12$ | $7^{\circ} 46' \pm 0^{\circ} 25'$ | 72 ± 43 |
| 7 | $1,04 \pm 0,16$ | $8^{\circ} 25' \pm 0^{\circ} 30'$ | 108 ± 57 |
| 8 | $1,20 \pm 0,25$ | $9^{\circ} 41' \pm 1^{\circ} 03'$ | 88 ± 99 |
| 9 | $1,04 \pm 0,16$ | $9^{\circ} 28' \pm 0^{\circ} 58'$ | 44 ± 75 |

$$\bar{U}_{\text{pes}} = 69,7 \cdot 10^{-4}; \quad \Delta e / \Delta i = 0,61 \pm 0,20.$$

TABELLA II.

(Traccia B: lunghezza 0,6 cm; inclinazione $8^{\circ}16'$; $Z \simeq 16$; $\bar{\alpha}_e < 4^{\circ}$).

| n | $\bar{\alpha}_e$ | $\omega \pm \Delta\omega$ | $(U \pm \Delta U) \cdot 10^4$ |
|-----|------------------|------------------------------------|-------------------------------|
| 1 | $3,78 \pm 0,68$ | $17^{\circ} 31' \pm 1^{\circ} 11'$ | 204 ± 253 |
| 2 | $1,59 \pm 0,27$ | $4^{\circ} 21' \pm 1^{\circ} 17'$ | 448 ± 95 |
| 3 | $3,39 \pm 0,59$ | $17^{\circ} 4' \pm 1^{\circ} 17'$ | 344 ± 236 |
| 4 | $1,53 \pm 0,22$ | $6^{\circ} 59' \pm 1^{\circ} 06'$ | 333 ± 86 |
| 5 | $2,09 \pm 0,92$ | $10^{\circ} 4' \pm 1^{\circ} 09'$ | 348 ± 119 |
| 6 | $1,62 \pm 0,25$ | $9^{\circ} 16' \pm 0^{\circ} 41'$ | 246 ± 86 |
| 7 | $3,13 \pm 0,43$ | $17^{\circ} 55' \pm 0^{\circ} 57'$ | -11 ± 140 |

$$\bar{U}_{\text{pes}} = 292 \cdot 10^{-4}; \quad \Delta e / \Delta i = 1,19 \pm 0,49.$$

TABELLA III.

(Traccia C: lunghezza 0,76 cm; inclinazione $20^{\circ}46'$; $Z \simeq 15$; $\alpha_e < 6^{\circ}$).

| n | $\bar{\alpha}_e$ | $\omega \pm \Delta\omega$ | $(U \pm \Delta U) \cdot 10^4$ |
|-----|------------------|------------------------------------|-------------------------------|
| 1 | $5,67 \pm 0,96$ | $18^{\circ} 31' \pm 2^{\circ} 41'$ | 725 ± 461 |
| 2 | $3,24 \pm 0,55$ | $13^{\circ} 25' \pm 1^{\circ} 27'$ | 467 ± 219 |
| 3 | $5,12 \pm 1,3$ | $18^{\circ} 14' \pm 1^{\circ} 40'$ | 579 ± 458 |
| 4 | $1,54 \pm 0,25$ | $9^{\circ} 51' \pm 1^{\circ} 55'$ | 186 ± 142 |
| 5 | $4,42 \pm 0,75$ | $15^{\circ} 58' \pm 2^{\circ} 16'$ | 608 ± 344 |
| 6 | $2,23 \pm 0,49$ | $8^{\circ} 43' \pm 1^{\circ} 45'$ | 475 ± 187 |
| 7 | $1,60 \pm 0,28$ | $7^{\circ} 25' \pm 1^{\circ} 59'$ | 305 ± 127 |
| 8 | $5,55 \pm 1,05$ | $17^{\circ} 49' \pm 3^{\circ} 13'$ | 723 ± 518 |
| 9 | $2,13 \pm 0,63$ | $11^{\circ} 31' \pm 1^{\circ} 49'$ | 283 ± 242 |

$$\bar{U}_{\text{pes}} = 351,8 \cdot 10^{-4}; \quad \Delta e / \Delta i = 0,73 \pm 0,24.$$

verso la consistenza esterna dei dati (cioè attraverso gli scarti della media). Il rapporto tra questi due errori, come ha mostrato BIRGE ⁽¹⁶⁾, se gli scarti attribuiti a ciascuna misura sono corretti, deve essere uguale all'unità, entro le fluttuazioni statistiche. Sotto ciascuna tabella è eseguito questo confronto, indicando con Δ_e e Δ_i rispettivamente l'errore di cui è affetta la media \bar{U} calcolato con la consistenza esterna e con la consistenza interna dei dati. L'accordo tra questi errori sembra essere abbastanza soddisfacente per cui possiamo concludere che è ragionevole assumere per l'errore su U il valore dato dalla (11).

I risultati ottenuti ci garantiscono quindi della riproducibilità della misura del parametro U , nelle condizioni sperimentali da noi stabilite.

5. - Conclusioni.

L'analisi da noi fatta dell'applicazione del metodo di misura della massa attraverso lo studio degli elettroni d'urto, conduce a concludere:

1) È possibile, nella maggior parte dei casi, conoscere a priori l'errore commesso nella determinazione della massa, anche nel caso in cui l'elettrone urtato sia uno solo. Nel 5% dei casi, tuttavia il valore della massa così ottenuto risulta errato per eccesso e quindi questo metodo può essere utile tutte le volte in cui è desiderabile conoscere un limite superiore della massa di una singola particella.

2) La frequenza degli urti, in cui si generano elettroni utilizzabili per la misura in questione, è in genere bassa ma non in maniera proibitiva. Infatti la frequenza degli elettroni d'urto con energia maggiore di 4 MeV è di circa 1 ogni 15 cm di traccia per particelle di carica unitaria e di 1 ogni 4 cm di traccia per particelle di carica 2.

3) La limitazione da noi fatta sull'energia dell'elettrone d'urto rende questo metodo applicabile solo a particelle di cui la massima energia trasferibile sia maggiore di 4 MeV, alle quali corrisponde un valore di U minore di 0,25 e cioè nel caso di particelle pesanti, una velocità β maggiore di 0,85 (mesoni di 175 MeV, protoni di 1,4 GeV). Per velocità molto maggiori, ossia per valori di U molto piccoli, sono utilizzabili solo elettroni emessi con angoli sempre più piccoli, cioè con energie sempre più vicine alla massima energia trasferibile, ed in questi casi il metodo diventa praticamente inutilizzabile a causa della bassa frequenza di tali eventi. Tenendo conto poi del fatto che l'intensità delle particelle della radiazione cosmica decresce rapidamente al

⁽¹⁶⁾ R. T. BIRGE: *Phys. Rev.*, **40**, 213 (1932).

crescere dell'energia, e che le misure sull'elettrone d'urto forniscono essenzialmente un valore della velocità, si ha che questa limitazione è meno restrittiva per protoni o per particelle di massa superiore a quella protonica che per mesoni.

Non sembra tuttavia irragionevole pensare ad una applicazione di questo metodo di misura delle masse allo studio della composizione isotopica di taluni nuclei della radiazione cosmica primaria. Infatti nel caso dei nuclei di elio, per esempio, una precisione del 25% sulla misura delle masse sarebbe sufficiente per valutare il rapporto ${}^3\text{He}/{}^4\text{He}$. Tuttavia va notato che a causa dell'elevata energia media delle particelle primarie occorrerà prendere in esame solo elettroni emessi sotto angoli piccoli ($< 10^\circ$) e quindi in media con un'energia maggiore di 15 MeV. La frequenza di questi eventi è circa 1 ogni 20 cm di traccia di elio.

Desideriamo ringraziare vivamente il prof. G. CORTINI e il prof. F. BRAMBILLA per le utili discussioni ed il prof. E. PANCINI e il prof. A. BORSELLINO per il costante incoraggiamento ed aiuto datoci nella presente ricerca.

APPENDICE I

Una valutazione grossolana della probabilità che ha un elettrone di essere scatterato di un angolo maggiore di θ_1 , da parte dell'atomo stesso al quale appartiene, si può fare facilmente, considerando una distribuzione uniforme di elettroni intorno al nucleo. Si ha, in tale ipotesi, che la probabilità che questo elettrone si trovi alla distanza b in db dal nucleo è data da:

$$(1) \quad \varphi(b) db = \frac{2\pi b db}{\pi r^2},$$

dove r è il raggio dell'atomo. Se questo elettrone acquista, ad un certo istante, un momento p , l'angolo θ di cui esso sarà deviato sarà: $\theta = 2Ze^2/bpc\beta$, e quindi la (1) diventa:

$$(2) \quad \varphi(\theta) d\theta = \frac{2(2Ze^2)^2 d\theta}{(rp\beta c)^2 \theta^3},$$

per cui la probabilità di avere una deviazione maggiore di un certo angolo θ_1 assegnato, sarà:

$$(3) \quad P(> \theta_1) \simeq \frac{2(2Ze^2)^2}{3r^2} \frac{1}{T^2\theta_1^2},$$

dove T è l'energia corrispondente al momento p .

Nel caso delle emulsioni nucleari G5 $\overline{Z^2} = 463$ e quindi $r = 1,91 \cdot 10^{-9}$ cm e la (3) per la nostra energia più bassa ($T=4$ MeV) e per $\theta_1 > 1^\circ$ dà: $P(> 1^\circ) \simeq 7 \cdot 10^{-4}$.

APPENDICE II

1. - Tenendo conto che nei casi che a noi interessano le perdite di energia per urto hanno effetto trascurabile, l'energia degli elettroni misurata, E , può differire da quella iniziale E_0 , sia per effetto degli errori statistici, sia a causa di un'eventuale perdita di energia per radiazione. Si noti che la misura è fatta su una lunghezza di traccia molto minore della lunghezza di radiazione e che, per tanto, è da aspettarsi che un'eventuale perdita per radiazione avvenga in un atto singolo.

Supposta l'energia dell'elettrone maggiore di 20 MeV (nel caso in cui $E < 20$ MeV i risultati ottenuti saranno errati per eccesso) si può calcolare la probabilità che un elettrone di energia E_0 perda per radiazione lungo un percorso Nt (t essendo la lunghezza della cella di misura) una frazione della sua energia compresa tra $(E_0 - E)/E_0 = y$ ed $[E_0 - (E + dE)]/E_0 = y + dy$, a mezzo della ⁽¹⁷⁾

$$(1) \quad W(y, Nt) = \frac{(\ln 1/y)^{bNt-1}}{\Gamma(bNt)} dy.$$

Tenendo conto solo degli urti che producono una variazione percentuale di energia più grande dell'errore statistico relativo $N^{-1/2}$, la percentuale di elettroni che hanno subito tale perdita sarà data dalla

$$(2) \quad \int_{\varepsilon}^{1-1/\sqrt{N}} W(y, Nt) dy,$$

dove $\varepsilon \neq 0$ rimane fissato dalla condizione che dopo la perdita di energia l'elettrone rimanga al minimo di ionizzazione.

Il valore di ε influisce poco sul valore dell'integrale (2) ed è stato da noi fissato ragionevolmente come eguale a 10^{-2} .

Il valore dell'integrale (2) risulta nel nostro caso di 12%, ma tale valore è certamente errato per eccesso perchè una data perdita di energia che avvenga in un solo atto radiativo influisce in modo diverso a seconda del punto della traccia in cui essa si è verificata, a causa del particolare modo in cui l'energia viene determinata dalla misura degli angoli di scattering. Inoltre, se la perdita è così grande da superare largamente gli errori statistici di misura degli angoli di scattering, essa potrà essere rivelata dall'analisi dei risultati delle misure.

2. - Per tener conto di quanto ora detto si può ragionare nel seguente modo.

La probabilità che in un tratto nt non ci sia perdita di energia per radia-

⁽¹⁷⁾ W. HEITLER: *The Quantum Theory of Radiation* (Oxford, 1954), pag. 378.

zione è data da:

$$1 - \int_E^{1-1/\bar{N}} W(y, nt) dy.$$

Una perdita di energia che avvenga solo nella $(n+1)$ -esima cella introduce un errore sulla media, al di fuori dell'errore statistico, solo se

$$E/E_0 < (N-n)/(N-n + \sqrt{N})$$

per cui la probabilità di una perdita importante nella $(n+1)$ -esima cella è data da:

$$A_n = \left[1 - \int_E^{1-1/\bar{N}} W(y, nt) dy \right] \int_E^{(N-n)/(N-n+1/\bar{N})} W(y, t) dy,$$

e la probabilità che questa perdita avvenga in una cella qualsiasi da: $B = \sum_n^{N-1} A_n$.

Il limite superiore $(N-n)/(N-n + \sqrt{N})$ è stato determinato tenendo conto che se $\sum_1^N \alpha_i/N \pm \Delta\bar{\alpha}$ è il valore dell'angolo medio di scattering $\bar{\alpha}$, misurato su tutta la traccia, nel caso che non vi sia perdita di energia, una perdita di energia che avviene nella $(n+1)$ -esima cella di misura, tale da aumentare i valori α_i delle rimanenti $N-n$ misure di un fattore X , introduce un errore solo se:

$$\frac{\sum_1^n \alpha_i + X \sum_{n+1}^N \alpha_i}{N} > \frac{\sum_1^N \alpha_i}{N} + \Delta\bar{\alpha},$$

da cui

$$(X-1)(N-n)\bar{\alpha}_{(N-n)} > \bar{\alpha}_{(N)}\sqrt{N}.$$

Perchè le medie $\bar{\alpha}_{(N)}$, ed $\bar{\alpha}_{(N-n)}$ fatte rispettivamente sulle N e sulle $N-n$ misure siano eguali entro l'errore statistico, deve essere

$$X > (\sqrt{N} + N - n)/(N - n),$$

ossia, essendo $X \cong E_0/E$,

$$E/E_0 < (N-n)/(\sqrt{N} + N - n).$$

Al valore dato dalla B occorre sottrarre la probabilità che un elettrone abbia subito nel tratto Nt una perdita considerevole di energia della quale ci si accorge dalle misure stesse. Questi elettroni sono tutti quelli per i quali si ha $E_{(N-n)}/E_{(n)} < 1 - 3\Delta[E_{(N-n)}/E_{(n)}]$, ove $E_{(n)}$ è l'energia ottenuta con le prime n celle ed $E_{(N-n)}$ quella ottenuta colle rimanenti $N-n$, cioè $E_{(N-n)}/E_{(n)} < 1/(1 + 3\sqrt{1/n + 1/(N-n)})$. La percentuale di questi elettroni è data da:

$$\sum_{10}^{N-10} \left\{ \left[1 - \int_E^{1-1/\bar{N}} W(y, nt) dy \right] \int_E^{1/(1+3\sqrt{1/n+1/(N-n)})} W(y, t) dy \right\}$$

essendo la somma estesa da 10 a $N-10$ in quanto per un numero di misure < 10 la determinazione dell'energia non ha alcun significato a causa dell'elevato errore statistico.

Ne deriva che la percentuale degli elettroni che introducono un errore è dato da:

$$P = \sum_0^{N-1} \left\{ \left[1 - \int_{\epsilon}^{1-1/\sqrt{N}} W(y, nt) dy \right] \int_{\epsilon}^{(N-n)/(N-n+1/\sqrt{N})} W(y, t) dy \right\} - \sum_{10}^{N-10} \left\{ \left[1 - \int_{\epsilon}^{1-1/\sqrt{N}} W(y, nt) dy \right] \int_{\epsilon}^{1/1+3\sqrt{1/n+1/(N-n)}} W(y, t) dy \right\}.$$

Il calcolo di P è stato fatto svolgendo gli integrali con la formula approssimata

$$\int_{y_1}^{y_2} \ln \left(\frac{1}{y} \right)^a \left(\frac{dy}{\ln 1/y} \right) = \left[\ln \frac{1}{y_1} \right]^a \int_{y_1}^{y_2} \frac{dy}{\ln 1/y}.$$

Nel caso dell'emulsione G_5 , si ha $b \simeq 4,5 \cdot 10^{-1} \text{ cm}^{-1}$, e per $N=50$, $t=30 \mu$, si ricava $P=5 \cdot 10^{-2}$.

SUMMARY

A method has been studied for the determination of the mass of minimum ionization particles in nuclear emulsions, in the case in which a knock-on electron is generated. In using the conservation theorems for energy and momentum one can determinate a function of the velocity of the particles, which is sensitive to variation of the velocity. Criteria have been given for the recognition of the knock-on electrons, and the various errors which influence the measurements of the energy and the angle of emission have been examined. The error on the determination of the mass of the incident particle has been calculated. An analysis of the distribution of the experimental values shows that the errors calculated according to criteria given by us, agree quite well with the experimental ones. Taking into account the frequency of these knock-on electrons, the limitations of this method as well as its possible application in the case of single events or in a systematic study of the isotopic composition of the primary cosmic radiation are discussed.

Zur quantentheoretischen Begründung der klassischen Physik.

I. — Dynamik der Gase und Flüssigkeiten.

H. KÜMMEL

Institut für theoretische Physik - Freie Universität - Berlin

(ricevuto il 14 Marzo 1955)

Zusammenfassung. — Die Gewinnung der klassischen Physik durch den Grenzübergang $\hbar \rightarrow 0$ ist nicht immer berechtigt. Daher wird dieser durch geeignete Voraussetzungen über die Zustände und Eigenwerte eines Systems von Teilchen ersetzt. Nur solche Observablen werden als makroskopisch zulässig betrachtet, die die von LUDWIG⁽¹⁾ formulierten Bedingungen erfüllen. Am Beispiel der Gasdynamik wird gezeigt, daß die klassische Physik eine Folge der Quantentheorie ist, indem die makroskopischen Observablen Dichte, Stromdichte und Spannungstensor eingeführt und deren Bewegungsgleichungen untersucht werden. Die Thermodynamik wird in einer folgenden Arbeit behandelt werden.

1. — Einleitung.

Beim Ringen um neue Erkenntnisse muß immer das bisher Bekannte als Ausgangspunkt dienen: So hat das Korrespondenzprinzip sich als *der* Wegweiser in der Quantentheorie erwiesen. Noch heute wird man — weil die Anfänger im Prinzip mit denselben Schwierigkeiten zu ringen haben wie die Entdecker — das Korrespondenzprinzip als didaktisches Hilfsmittel im Unterricht benutzen müssen; als solches wird es immer seine Berechtigung haben.

Es ist klar, daß — von der klassischen Physik herkommend — unter Verwendung der in ihr allein nicht verständlichen Erfahrungen die Quantentheorie *fast* zwingend erschlossen werden kann. Die Algebraisierung des Zusammenhanges beider Theorien durch FALK⁽²⁾ zeigt dies besonders deutlich. Allein

⁽¹⁾ G. LUDWIG: *Zeits. f. Phys.*, **135**, 483 (1953); *Die Grundlagen der Quantenmechanik* (Berlin, Göttingen, Heidelberg, 1954), Kap. V und Anhang IV.

⁽²⁾ G. FALK: *Zeits. f. Phys.*, **135**, 431 (1953).

dieser Umstand darf nicht darüber hinwegtäuschen, daß es allenfalls gelingt, Strukturverwandtschaften festzustellen, daß es aber niemals möglich sein kann, beide Theorien für sich zu betrachten. Die klassische Theorie muß sicherlich eine Folge der Quantentheorie sein. Was heißt hier aber: Eine Folge? Doch nichts anderes als: Durch das Zusammenwirken sehr vieler Teilchen unter Bedingungen, die i. A. sehr erheblich von denen eines mikroskopischen Experimentes abweichen, entsteht das makroskopische Erscheinungsbild. Dies äußert sich vor allem darin,

- a) daß *sehr viele* Teilchen vorhanden sind,
- b) daß der Zustand aller Teilchen eines makroskopischen Systems *nicht einmal optimal* im Sinne der Quantenmechanik bekannt ist, daß also ein Gemisch vorliegt,
- c) daß im wesentlichen Energien eine Rolle spielen, die erheblich über den sonst in der Quantenmechanik betrachteten liegen.

Das alles muß so wirken, daß makroskopisch das klassische Verhalten (mit makroskopischer Genauigkeit) resultiert.

Wie jedoch soll man von der Quantenmechanik her diese Ergebnisse erhalten, und zwar *ohne daß Einzelheiten der Mikrostruktur wesentlich eingehen*? Es soll ja zum Beispiel die genaue Form des Wechselwirkungspotentials der Moleküle eines Gases die *makroskopischen Prinzipien* nicht beeinflussen.

Offenbar kann man nicht einfach $\hbar = 0$ setzen, wie das so oft in Lehrbüchern getan wird; dies *kann* gut gehen, wenn man geeignete kanonische Transformationen durchführt. Man erhält so z.B. ⁽³⁾ die Hamiltonfunktion

$$H\left(\frac{\partial S}{\partial q} + \frac{\hbar}{i} \frac{\partial}{\partial q}, q\right),$$

was für $\hbar \rightarrow 0$ klassisches Verhalten für ein Teilchen bedeutet. Da aber — wie man weiß — der Operator $\partial/\partial q$ i.A. nicht als klein betrachtet werden kann (mindestens ist eine spezielle Voraussetzung über die Wellenfunktion erforderlich), ist es notwendig, die Verhältnisse ohne einen solchen Grenzübergang zu studieren. Und was heißt $\hbar \rightarrow 0$; wogegen soll \hbar klein sein? Wir haben vorerst nicht im Sinn, die durch die Quantentheorie hervorgerufenen Korrekturen zu studieren, sondern wollen das « primitive » makroskopische Verhalten gewinnen. Es liegt nahe, daß ein gewisser Zusammenhang mit dem eben angedeuteten Verfahren besteht, weil das eventuelle Resultieren der klassischen Physik mit $\hbar \rightarrow 0$ (also das Funktionieren des Korrespondenzprinzips) sicherlich seine

⁽³⁾ P. A. M. DIRAC: *Principles of Quantum Mechanics*, 3. Aufl. (Oxford, 1947), S. 122, u.v.a..

tieferliegenden Gründe hat. In der WKB-Methode ⁽⁴⁾ hat ja die Entwicklung nach \hbar praktische Bedeutung erlangt. Auch in der Flüssigkeitstheorie hat man erfolgreich davon Gebrauch gemacht ⁽⁵⁾.

Die an vielen Beispielen bestätigte Erscheinung, daß für hohe Energien die klassische Physik (schon für ein Einzelteilchen) gilt, ist allgemein nicht richtig. Zwar hat ein System *fester* hoher Energie oft eine klassische Verteilung der Aufenthaltswahrscheinlichkeit (am besten untersucht ist das Beispiel des harmonischen Oscillators). Wenn Teilchen *frei* sein können (wenn also die Potentialmulde nicht unendlich hoch ist), wird die Sache aber in charakteristischer Weise anders: Bei *fester* hoher Energie haben wir (praktisch) ebene Wellen, bei einer gewissen *Energieunschärfe* können wir durch passende Überlagerung (Wellenpaket) ein klassisches Partikel erhalten, das für sehr hohe Energien praktisch nicht zerfließt. Dabei ist von dem Grenzübergang $\hbar \rightarrow 0$ nicht die Rede. Das wesentliche ist, daß *klassisches Verhalten* die *Energieunschärfe* erfordert, daß aber weder $\hbar \rightarrow 0$ noch $E \rightarrow \infty$ (oder Masse $m \rightarrow \infty$) ausreicht.

Diese eben angestellten Überlegungen gelten ebenso für ein « leichtes » Teilchen hoher Energie wie für ein « schweres » Teilchen niedriger Energie, also auch für den Schwerpunkt eines Systems von (praktisch) fest aneinander gebundenen Teilchen.

Wie aber wird man die Gas- und Flüssigkeitsdynamik beschreiben können, wo doch offensichtlich von der Schwerpunktbewegung der Teilchen eines herausgegriffenen Volumenelementes nach der Quantentheorie nicht gesprochen werden kann? Und was heißt: Die Teilchen eines Volumenelementes? Hiermit hängt die Frage zusammen, was die Geschwindigkeitsverteilungsfunktion der Teilchen an einem Ort für eine quantentheoretische Bedeutung hat. Wohl kann man eine Funktion angeben, die bereits für ein einzelnes Teilchen den Erwartungswert der Geschwindigkeit liefert. Diese ist mit der WIGNER'schen Verteilungsfunktion identisch. VAN KAMPEN ⁽⁶⁾ hat bereits darauf hingewiesen, daß diese zur Gewinnung der Statistik ungeeignet ist, weil sie nur die statistischen Aussagen der *Quantentheorie* und keine Aussagen über makroskopische Beobachtungen liefert. Wir werden am Schluß dieser Arbeit hierauf zurückkommen.

Daß man mit allen diesen Betrachtungen nicht leicht weiterkommt, ist ganz natürlich. Was man makroskopisch beobachtet, ist ja etwas ganz anderes als eine mikroskopische Größe. Wir müssen also den Begriff der « *makroskopischen Observablen* » einführen und untersuchen, welche aller möglichen Obser-

⁽⁴⁾ Zahlreiche Literaturangaben z. B. bei: L. SCHIFF: *Quantum Mechanics* (New York, Toronto, London, 1949), S. 178.

⁽⁵⁾ H. S. GREEN: *Molecular Theory of fluids* (Amsterdam, 1952), Kap. 9.

⁽⁶⁾ VAN KAMPEN: *Physica*, 20, 603 (1954).

vablen auszuwählen sind. Dies hat LUDWIG ⁽¹⁾ bereits in aller Konsequenz getan: Es sind an solche Observablen ganz bestimmte Forderungen zu stellen (siehe unten, 2·2). Diese Forderungen resultieren aus der Gültigkeit des Ergodensatzes. Nur solche Größen sind makroskopisch bedeutsam, die für $t \rightarrow \infty$ einem konstanten Wert zustreben, der das Gleichgewicht bestimmt (selbstverständlich handelt es sich hierbei um die inneren Variablen; für äußere braucht dies nicht zu gelten, denn für diese ist die Gültigkeit der klassischen Mechanik kein Problem; Beispiel: Schwerpunkt des Gesamtsystems). Ferner « enthält » (in einem Gas bzw. einer Flüssigkeit) jedes makroskopischen Untersuchungen zugängliche Volumenelement immer noch sehr viele Teilchen und schließlich ist jede makroskopische Beobachtung eine Mittelung über ein kleines Zeitintervall, sodaß sich die statistischen Schwankungen nicht bemerkbar machen (genauer: sich nur bei großen Abweichungen vom Gleichgewicht bemerkbar machen; z.B. BROWN'sche Bewegung bei hinreichend kleiner Wahl, Diffusion bei mittelgroßer Wahl und Gleichgewichtszustände bei unendlich großer Wahl des Zeitintervalles).

LUDWIG ⁽¹⁾ hat das Erreichen des Gleichgewichtes im Zusammenhang mit den makroskopischen Auswirkungen mikroskopischer Vorgänge bei einer Messung eingehend untersucht mit der sog. « nichtdynamischen Methode », d.h. durch Mittelung über sehr große Zeiten. Neuerdings hat VAN KAMPEN ⁽⁶⁾ (offenbar ohne Kenntnis der LUDWIG'schen Untersuchungen) mit der « dynamischen Methode » darüber-hinausgehende Resultate erhalten. Er zeigt, daß die zur Grundlegung der statistischen Mechanik notwendigen Gleichungen für die Besetzungszahlen

$$\dot{P}_i = \sum_j (w_{ij} p_j - w_{ji} p_i)$$

mit

$$w_{ij} = w_{ji}$$

sich quantenmechanisch (ohne die sehr anfechtbare Störungsrechnung) begründen lassen. Er erhält damit die ONSAGER'sche phänomenologische irreversible Thermodynamik ⁽⁷⁾. Die p_i hängen sehr kompliziert mit den makroskopischen Observablen zusammen und es besteht wenig Hoffnung, diesen Zusammenhang im Einzelfall zu durchschauen.

Bei beiden erwähnten Arbeiten blieb aber ein ganzer Fragenkomplex offen: Gibt es solche makroskopischen Observablen und wie leitet man sie aus den mikroskopischen her?

Da die Antwort auf diese Frage leicht zu finden sein wird, stellt sich die nächste: Welche Bedeutung haben die so gewonnenen makroskopischen Observablen; d.h. welchen beobachtbaren Größen entsprechen sie?

⁽⁷⁾ z.B. S. R. DE GROOT: *Thermodynamics of Irreversible Processes* (Amsterdam, 1954).

Diese Frage beantwortet sich durch die Kenntnis der Bewegungsgleichungen, deren Gewinnung etwas Rechenaufwand erfordert. Es ist klar, daß bei Kenntnis der Antworten auf diese Fragen auch die Probleme der irreversiblen Thermodynamik in Angriff genommen werden können; dazu gehören also z.B. Strömungsvorgänge im allgemeinsten Sinn (einschließlich Diffusion, Thermodiffusion usw.), die wir erst vollständig beschreiben können, wenn alle eingehenden Größen mikroskopisch begründet sind.

Übersicht: In dem folgenden ersten Teil der Untersuchungen zu diesem Problemkreis wird die «mechanische» Seite der makroskopischen Physik analysiert. Zunächst werden die Voraussetzungen über Energieeigenwerte und die Eigenfunktion formuliert und die makroskopischen Observablen definiert. Dann werden spezielle Observable (Teilchendichte, Stromdichte und Spannungstensor) eingeführt. Da wir hier nur die Dynamik betrachten, interessiert vorerst nur die geordnete Bewegung des Systems. Wir nehmen daher an, daß es sich in einem bestimmten (makroskopischen) Energiezustand befindet. Ferner sollen die Wechselwirkungskräfte nicht zu lange Reichweiten besitzen und so schwach sein, daß noch merkliche innere Bewegungen des Systems möglich sind. Damit haben wir das Problem der Gas- oder Hydrodynamik. Es wird gezeigt, daß genau die bekannten Euler'schen klassischen Gleichungen gelten. Die eingeführten Größen stehen in engem Zusammenhang mit den klassischen (mikroskopisch begründeten, siehe GREEN ⁽⁵⁾). Zum Schluß wird unsere Theorie noch mit der üblichen Methode der Entwicklung nach \hbar verglichen.

Die thermodynamische Seite der Theorie läßt sich ganz nach denselben Grundsätzen entwickeln; sie wird demnächst in einer weiteren Arbeit folgen. Es besteht die Hoffnung für die Theorie des He II neue wesentliche Gesichtspunkte zu gewinnen.

Auch die Aufgabe, gerade das Abweichen vom klassischen Verhalten zu untersuchen, stellt sich. Schließlich wäre auch eine interessante Frage, warum man mit klassischen Vorstellungen (Boltzmann'sche Stoßgleichung, H -Theorem) bei geringen Dichten so erfolgreich ist; warum also manchmal eine klassische *mikroskopische* Theorie das Richtige liefert. Gerade in diesem Falle, in dem man mit klassischen punktförmigen Teilchen arbeitet, muß der Erfolg verwundern.

2. – Makroskopische und mikroskopische Größen.

2.1. *Energiezustände und Eigenwerte.* – Von LUDWIG wird für die Energieeigenwerte ε_n des Systems gefordert ⁽⁸⁾:

⁽⁸⁾ G. LUDWIG: *l. c.*, S. 142 ff.; auch von VAN KAMPEN müssen diese Voraussetzungen gemacht werden; der Verf. formuliert sie z.T. allerdings nicht explizit.

- a) Sie sind diskret im Bereiche $0 \leq \varepsilon_n \leq \varepsilon_{\max}$, liegen jedoch sehr dicht beieinander. (Jede makroskopische Messung erfolge mit einer Meßgenauigkeit $\Delta\varepsilon$; dann sollen in $\Delta\varepsilon$ immer noch sehr viele Eigenwerte ε_n liegen).
- b) Sie sind unregelmäßig (statistisch) über das Intervall $(0, \varepsilon_{\max})$ verteilt. Genauer: Der Fall $\varepsilon_n - \varepsilon_m = 0$ für $n \neq m$ soll nur sehr selten vorkommen (die Zahl der Resonanzen $\varepsilon_n - \varepsilon_m = 0$ mit $E_1 \leq \varepsilon_n$, $\varepsilon_m \leq E_2$ soll klein sein gegen die Zahl der Eigenwerte in (E_1, E_2)).

Zusätzlich ist noch die Annahme gemacht, daß keine wesentliche Entartung vorliegt (sie darf nur zufällig und nicht zu stark sein). Die Bedingung a) ist für jedes System von Teilchen erfüllt, die in einem Kasten eingeschlossen sind. b) ist die für die Gültigkeit des Ergodensatzes entscheidende Bedingung und entspricht etwa der sog. Unordnungsannahme von PAULI ⁽⁹⁾.

Der Beweis des Ergodensatzes ist bisher in klassischer und Quantentheorie nur insofern gelungen, als man die Bedingungen angeben kann, unter welchen er gilt. Alle Versuche, wenigstens ein Modell zu konstruieren, das ihn erfüllt, sind gescheitert. Dagegen gelingt es leicht, Gegenbeispiele zu finden. Das ist kein Wunder: Gerade die Unordnung der Eigenwerte schließt aus, daß es eine vollständige Berechnungsmethode für sie gibt. Denn nur solche Eigenwerte sind berechenbar, die einem einfachen Gesetz gehorchen — diese aber widersprechen naturgemäß Forderung b).

Wir werden uns also damit begnügen, die Eigenwerte als bekannt anzunehmen und die Ergebnisse zu gewinnen, die ohne nähere Kenntnis derselben erhalten werden können. Man kann erwarten, daß es garnicht so sehr auf diese ankommt (Alle Tieftemperaturanomalien freilich weisen auf Fälle hin, bei denen sehr wesentlich die Termelage der niedrigsten Energien eingeht).

Mit der Unkenntnis der ε_n ist eine Unkenntnis der ψ_n -Funktion verknüpft. Hier müssen wir uns mit weitgehend plausiblen Annahmen begnügen: Für alle ε_n (die bei «normalen» Temperaturen wesentlich sind), soll die Schrödingergleichung des N -Teilchen-Problems

$$(1) \quad H\psi_n = \varepsilon_n\psi_n$$

mit

$$(2) \quad H = -\frac{\hbar^2}{2m} \sum_i \Delta_i + \sum_{i < k}^N V_{ik}(\mathbf{r}_i, \mathbf{r}_k) + V^{(a)}(\mathbf{r}_e),$$

\mathbf{r}_e = Koordinaten der N -Teilchen

V_{ik} = Wechselwirkungspotential zweier Teilchen

$V^{(a)}$ = äußeres Potential

⁽⁹⁾ W. PAULI: *Sommerfeld-Festschrift* (1928).

durch den bekannten Ansatz

$$(3) \quad \psi_n(\mathbf{r}_i) = A_n(\mathbf{r}_i) \exp \left[(i/\hbar) W_n(\mathbf{r}_i) \right],$$

mit reellen A_n und W_n gelöst werden, wenn wir vorerst das Pauli-Prinzip außeracht lassen. Damit geht ⁽¹⁰⁾ (1) über in

$$(4) \quad \begin{cases} (a) & 2m(\varepsilon_n - V) = \sum_i (W_{n|v})^2 \\ (b) & \sum_i W_{n|v} + 2A_n \sum_i A_{n|v} W_{n|v} = 0, \end{cases}$$

mit den Abkürzungen

$$(5) \quad \begin{cases} W_{n|v} = \frac{\partial W}{\partial x_v(i)} & v = 1, 2, 3 \\ & i = 1 \dots N \\ V = \sum_{ik} V_{ik} + V^{(a)} \end{cases}$$

(dabei ist das Glied $-(\hbar^2/2m) \sum_i A_i A_n / A_n$ weggelassen; s. unten). Die Gleichung (4a) ist die Eikonalgleichung der Quantentheorie und liefert den Zugang zur geometrischen Wellenmechanik, d.h. zur klassischen Physik; (4b) ist die Kontinuitätsgleichung und ist identisch erfüllt ⁽¹¹⁾. Die Näherungsannahmen beziehen sich alle auf eine « kleinste makroskopische Länge » ΔX_v , die folgende Bedingungen für alle ε_n an allen Stellen erfüllen soll:

$$(6) \quad 1.) \quad \Delta X_v \gg \lambda_n^{(i)} \equiv \frac{\hbar}{|W_{n|v}|},$$

$\lambda_n^{(i)}$ = lokale Wellenlänge;

$$(7) \quad 2.) \quad \frac{W_{n|v|k}^\mu}{W_{n|v|k}} \ll \frac{1}{\Delta X_v};$$

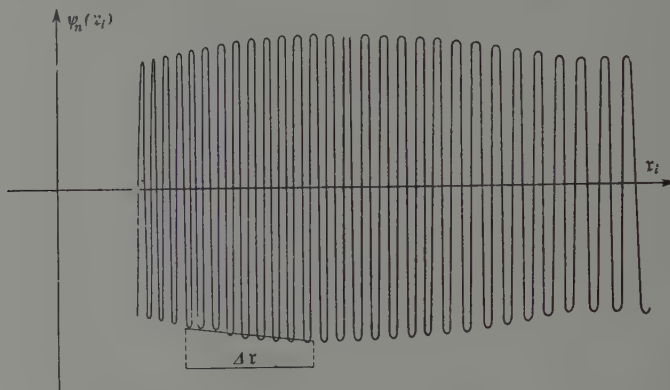
$$(8) \quad 3.) \quad \frac{A_{n|v|k}}{A_n} \ll \frac{1}{\Delta X_v}.$$

Produkte von Größen $W_{n|v|k}^\mu$ und $A_{n|v|k}$ sowie alle höheren Ableitungen werden konsequenterweise als von zweiter bzw. höherer Ordnung klein weggelassen, während die Größen selbst noch immer mitgenommen werden.

⁽¹⁰⁾ Dies Verfahren ist das übliche zur Auffindung des klassischen Verhaltens angewandte, s. DIRAC: *l. c.*; es ist auch Ausgangspunkt der sog. W. K. B.-Näherung.

⁽¹¹⁾ Die zwei Gleichungen (4) sind den beiden Schrödingergleichungen (1) für Real- und Imaginärteil von ψ_n äquivalent.

Die Bedeutung dieser Voraussetzungen ist klar: Im Intervall ΔX_ν sollen immer noch sehr viele praktisch gleiche Wellenlängen der ψ -Funktion liegen und die Amplitude sich nicht wesentlich (nur linear) ändern (s. Abb.). Die Gl. (4a) zusammen mit der Näherung (7) bedeuten eine Einschränkung für das Potential V : Im wesentlichen ist $(1/2m)(W_{n|v})^2$ die kinetische Energie des i -ten Teilchens. Daher kann man $V_{|i} \ll E_{kin}/\Delta X_\mu$ folgern, oder etwa $V \ll E_{kin}$. Diese Annahme steckt auch in der klassischen Molekulartheorie der Flüssigkeiten und Gase (s. GREEN⁽⁵⁾). Die kurzreichweitigen steilen Abstossungspotentiale sind



also nicht berücksichtigt; unsere Theorie wird daher das Verhalten der Gase richtiger beschreiben als das der Flüssigkeiten und um so besser stimmen, je höher die Temperatur ist. (Die hier gemachten Einschränkungen lassen sich erheblich abschwächen. Dies kann für die Begründung des hydrodynamischen Kernmodelles wesentlich werden; Untersuchungen hierüber sind im Gange.)

Einige wenige ψ_n -Funktionen brauchen sich in diese Abschätzungen nicht einzuordnen; die nachfolgenden Überlegungen werden davon nicht betroffen.

Man schätzt leicht ab, daß bei mittleren Temperaturen die Wellenlänge λ bei 10^{-8} cm liegt, wenn man freie Teilchen annimmt. Dies wird sich bei Einschalten der Wechselwirkung nicht wesentlich ändern; Die Zahl der Knoten ist nach bekannten Oszillationstheoremen gleich n . Über das ganze Volumen darf sich sowohl λ wie A_n beliebig ändern. Das Pauliprinzip berücksichtigen wir durch den Ansatz

$$(9) \quad \psi_n^\pm(\mathbf{r}_e) = \frac{1}{N!} \sum_p (\pm 1)^p A_n(\mathbf{r}_1 \dots \mathbf{r}_N) \exp \left[(i/\hbar) W_n(\mathbf{r}_1 \dots \mathbf{r}_N) \right].$$

(Summe über alle Permutationen der Koordinaten $\mathbf{r}_1 \dots \mathbf{r}_N$; ± 1 je nachdem, ob die Teilchen der Bose- oder Fermistatistik zu unterwerfen sind). Wir denken uns ferner ψ_n so normiert, daß

$$(10) \quad (\psi_n, \psi_n) = N$$

ist. Die Gleichungen (4) sind dann entsprechend zu ändern.

2.2. *Makroskopische Observablen.* – Die bisherigen Überlegungen sind (abgesehen von der trivialen Verallgemeinerung auf mehrere Teilchen) mehr als eine übliche Betrachtung der geometrischen Wellenmechanik, d.h. des klassischen Grenzfalles: Erstens gibt die Eikonalgleichung (4a) nur die Bahnen der Bewegung. Sie sagt noch nichts aus über die Aufenthaltswahrscheinlichkeit. Diese erhält man erst in Verbindung mit der Kontinuitätsgleichung (4b); letztere aber beschreibt immer noch die *quantenmechanische* Teilchendichte, besitzt also direkt keine klassische Bedeutung.

Wie in der Einleitung bemerkt, ist der Begriff der makroskopischen Beobachtung noch einzuführen. Unsere Aufgabe ist es also, die obigen Vernachlässigungen in die noch zu definierenden makroskopischen Observablen einzubeziehen. Erst beides zusammen liefert die klassische Physik.

Wir referieren in diesem § zunächst von LUDWIG ⁽¹⁾ erhaltene Ergebnisse.

Wie einleitend schon erwähnt, führt die Forderung nach der Gültigkeit des Ergodensatzes zu gewissen Forderungen auch an die Observablen. Unter allen möglichen Observablen haben nur diejenigen makroskopische Bedeutung, die folgende Bedingungen erfüllen (Wir wollen diese dann «makroskopische Observable» oder m. O. nennen):

- 1.) Jede m.O. ist «beinahe diagonal». «Fast alle» Matrixelemente der Energiedarstellung liegen in der Nähe der Diagonalen mit der mittleren Breite $\Delta\varepsilon$ ⁽¹²⁾:

$$(\varepsilon_n | O_t | \varepsilon_m) = 0 \quad \text{für} \quad |\varepsilon_n - \varepsilon_m| > \Delta\varepsilon.$$

- 2.) Es gibt immer sehr viele Matrixelemente gleicher Größenordnung und O variiert von einem $(\varepsilon_n, \varepsilon_m)$ -Wertepaar zum nächsten «fast immer» um nur kleine Beträge.

Zu Forderung 1: Da jede makroskopische Messung eine Mittelung über eine (makroskopisch verschwindend kleine) Zeit $\Delta\tau$ impliziert, kann man Operatoren durch Multiplikation mit z.B. einer Gaussfunktion $\exp[-(\Delta\tau^2/4\hbar^2)(\varepsilon_n - \varepsilon_m)^2]$ zu m.O. machen. Umgekehrt muß ein makroskopischer Operator etwa eine derartige Abhängigkeit von $|\varepsilon_n - \varepsilon_m|$ haben. Daher hängt das kleinste *makroskopische* Zeitintervall $\Delta\tau$ mit $\Delta\varepsilon$ über

$$(11) \quad \Delta\varepsilon \Delta\tau \gg \hbar$$

zusammen.

Die Größe

$$(12) \quad (\varepsilon_n | O_t | \varepsilon_m) = \exp[(i/\hbar)(\varepsilon_n - \varepsilon_m)(t - \tau)] (\varepsilon_n | O_0 | \varepsilon_m)$$

⁽¹²⁾ $\Delta\varepsilon$ ist die auf S. 1062 eingeführte makroskopische Energieungenauigkeit.

können wir nach Forderung 2 als Funktion von

$$\varepsilon_{nm} = \frac{1}{2}(\varepsilon_n + \varepsilon_m)$$

in dem Intervall $(\varepsilon, \Delta\varepsilon)$ als von ε_{nm} unabhängig ansehen. Führen wir noch

$$\frac{1}{\hbar}(\varepsilon_n - \varepsilon_m) = \omega_{nm}$$

ein, so erlauben uns die oben definierten Eigenschaften der m.O.

$$(13) \quad (\varepsilon_n | O_t | \varepsilon_m) = O(\varepsilon, \omega) \exp[i\omega(t - \tau)]$$

zu schreiben, und als « m.O. bei der festen Energie ε » die Größen

$$(14) \quad O_m(\varepsilon, t - \tau) = \frac{1}{\sigma(\varepsilon)} \int \varrho(\varepsilon, \omega) O(\varepsilon, \omega) \exp[i\omega(t - \tau)] d\omega,$$

einzuführen. Dabei ist

$$(14a) \quad \varrho(\varepsilon, \omega) = \sigma(\varepsilon) \delta(\omega) + \tau(\varepsilon, \omega)$$

die Dichte der Energieterme an der Stelle (ε, ω) . Es läßt sich zeigen ⁽¹³⁾, daß alle m.O. miteinander vertauschbar sind und daß die $O_m(\varepsilon, -\tau_1)$ die Eigenwerte der Observablen O darstellen, wenn sich das System in der Energie ε befindet. Die Eigenfunktionen sind Funktionen $\delta(\tau - \tau_1)$ d.i. eine δ -Funktion mit einer Breite $\gg \hbar/\Delta\varepsilon$. Wenn das System ein « Gemisch » ist, so sind alle Überlegungen entsprechend zu verallgemeinern. Der Erwartungswert der Observablen O ist dann

$$(15) \quad \int w^*(\varepsilon, \omega) \varrho(\varepsilon, \omega) O(\varepsilon, \omega) \exp[i\omega\tau_1] d\omega d\varepsilon$$

mit dem « Dichteoperator »

$$(16) \quad (\varepsilon_n | W_t | \varepsilon_m) = w(\varepsilon, \omega).$$

Produkte von makroskopischen Observablen sind wieder makroskopische Observable:

$$(17) \quad \sum_{\varepsilon} (\varepsilon_n | O_t | \varepsilon_e) (\varepsilon_e | O'_t | \varepsilon_m) \approx \sum_{\varepsilon} O_0(\varepsilon, \omega_{ne}) O'_0(\varepsilon, \omega_{em}) \exp[i\omega_{nm}(t - \tau)] \approx \\ \approx \int \varrho(\varepsilon, \omega') O(\varepsilon, \omega') O'(\varepsilon, \omega - \omega') \exp[i\omega(t - \tau)] d\omega'.$$

⁽¹³⁾ G. LUDWIG: l. c., Anh. IV.

Mit

$$(18) \quad [OO'](\varepsilon, \omega) = \int \varrho(\varepsilon, \omega') O(\varepsilon, \omega') O'(\varepsilon, \omega - \omega') d\omega'$$

wird also

$$(19) \quad OO' \rightarrow \frac{1}{\sigma(\varepsilon)} \int \varrho(\varepsilon, \omega) [OO'](\varepsilon, \omega) \exp[i\omega(t - \tau)] d\omega.$$

Dies ist andererseits offenbar

$$= \frac{1}{\sigma(\varepsilon)} \int \varrho(\varepsilon, \omega) O(\varepsilon, \omega) \exp[i\omega(t - \tau)] d\omega \cdot \frac{1}{\sigma(\varepsilon)} \int \varrho(\varepsilon, \omega') O'(\varepsilon, \omega') \exp[i\omega'(t - \tau)] d\omega'.$$

2.3. *Makroskopische Differentiation und Integration.* – Die bisherigen Überlegungen legen es nahe, eine «makroskopische Differentiation in Bezug auf das Teilchen i » durch

$$(20) \quad \frac{\Delta^{(i)} f(\mathbf{r}_i)}{\Delta X_\nu} \equiv f_{i\nu}(\mathbf{r}_i) \equiv \frac{f(\mathbf{r}_i + \Delta X_\nu) - f(\mathbf{r}_i)}{\Delta X_\nu}$$

zu definieren.

Dabei ist ΔX_ν die oben eingeführte «kleinste makroskopische Länge».

Als «makroskopische Differentiation» bezeichnen wir die Operation

$$(21) \quad \frac{\Delta f(\mathbf{r})}{\Delta X_\nu} \equiv f_{\nu}(\mathbf{r}) = \frac{f(\mathbf{r} + \Delta x_\nu, \dots, \mathbf{r} + \Delta X_\nu) - f(\mathbf{r}, \dots, \mathbf{r})}{\Delta X_\nu}.$$

Wenn $f(\mathbf{r}_i)$ eine (in allen $3N$ Variablen) langsam veränderliche Funktion ist, so gilt

$$(22) \quad f_{i\nu}(\mathbf{r}_i) = f_{\nu}(\mathbf{r}_i)$$

und

$$(23) \quad f_{\nu}(\mathbf{r}_i) |_{\mathbf{r}_i = \mathbf{r}} = \sum_i^N f_{i\nu}(\mathbf{r}_i) |_{\mathbf{r}_i = \mathbf{r}}.$$

Als «makroskopische Integration» bezeichnen wir die Summation über alle durch $\Delta x_1 \Delta x_2 \Delta x_3$ charakterisierten Raumzellen $\Delta V(\mathfrak{R}_e)$ an der makroskopischen Stelle \mathfrak{R}_e :

$$(24) \quad \sum_{\mathfrak{R}_e} \Delta V(\mathfrak{R}_e) f(\mathbf{r}_i + \mathfrak{R}_e) = \int_V f(\mathbf{r}_i) d\mathfrak{R}.$$

Ein wichtiges Beispiel für die Funktion f ist das Integral (mit $W_{mn} = W_m - W_n$)

$$(25) \quad F(W_{mn}, \mathfrak{R}) \equiv \int_{\Delta V(\mathfrak{R})} \exp[(i/\hbar) W_{mn}] d\mathbf{r}_1,$$

integriert über das «kleinste makroskopische Volumen» an der makroskopischen Stelle \mathfrak{R} . W_n sind z.B. die in 2.1 eingeführten Phasenfunktionen. Es ist wegen (7)

$$\begin{aligned} F_{\mathbf{1}\nu}(W_{mn}, \mathfrak{R}) &= \frac{\int_{\Delta V(\mathfrak{R}) + \Delta X_\nu} \exp[(i/\hbar) W_{mn}] d\mathbf{r}_1 - \int_{\Delta V(\mathfrak{R})} \exp[(i/\hbar) W_{mn}] d\mathbf{r}_1}{\Delta X_\nu} = \\ &= \frac{1}{\Delta X_\nu} \int_{\Delta V(\mathfrak{R})} \exp\left[\frac{i}{\hbar} W_{mn}(\mathfrak{R} + \mathbf{r}_1 + \Delta X_\nu)\right] - \exp\left[\frac{i}{\hbar} W_{mn}(\mathfrak{R} + \mathbf{r}_1)\right] d\mathbf{r}_1 = \\ &= \frac{1}{\Delta X_\nu} \int_{\Delta V(\mathfrak{R})} \left(\exp\left[\frac{i}{\hbar} W_{mn|_{\mathbf{1}\nu}} \Delta X_\nu\right] - 1 \right) \exp\left[\frac{i}{\hbar} W_{mn}(\mathfrak{R} + \mathbf{r}_1)\right] d\mathbf{r}_1. \end{aligned}$$

Also:

$$(26) \quad F_{\mathbf{1}\nu}(W_{mn}, \mathfrak{R}) = \frac{\exp[(i/\hbar) W_{mn|_{\mathbf{1}\nu}}(\mathfrak{R}) \Delta X_\nu] - 1}{\Delta X_\nu} F(W_{mn}, \mathfrak{R}).$$

Nehmen wir nun an, daß $W_{mn|_{\mathbf{1}\nu}} = W_{m|_{\mathbf{1}\nu}} - W_{n|_{\mathbf{1}\nu}}$ so klein ist, daß

$$(27) \quad |W_{mn|_{\mathbf{1}\nu}}| \Delta x_\nu \ll \hbar$$

ist, dann wird (weil $W_{mn|_{\mathbf{1}\nu}}$ in ΔX_ν langsam veränderlich mit (22))

$$(28) \quad F_{\mathbf{1}\nu}(W_{mn}, \mathfrak{R}) = \frac{i}{\hbar} W_{mn|_{\mathbf{1}\nu}} F(W_{mn}, \mathfrak{R}),$$

eine Beziehung, die wir später brauchen.

Wir wollen hier noch abschätzen, was die Bedingung (27) bedeutet und ob sie mit (6) verträglich ist. (27) bedeutet

$$\Delta X_\nu \ll \frac{\hbar}{|W_{mn|_{\mathbf{1}\nu}}|} = \frac{1}{2\pi} \frac{\lambda_m^{(1)} \lambda_n^{(1)}}{|\lambda_m^{(1)} - \lambda_n^{(1)}|} \approx \sqrt{\frac{\varepsilon^{(1)}}{m}} \cdot \frac{\hbar}{\Delta \varepsilon^{(1)}}$$

(wenn wir $\varepsilon_n^{(i)} = \hbar^2/2m\lambda_n^{(i)2}$ setzen, also freie Teilchen nehmen und $\Delta \varepsilon^{(i)} =$

$= h^2/2m |1/\lambda_m^{(i)2} - 1/\lambda_n^{(i)2}|$ abkürzen). Nun ist die Gesamtenergie

$$\varepsilon = \sum_i^N \varepsilon^{(i)} \sim N \varepsilon^{(i)}; \quad \Delta' \varepsilon \cong N \Delta \varepsilon^{(i)}$$

also (mit $\lambda^{(i)} = (\lambda_n^{(i)} + \lambda_m^{(i)})/2$)

$$(29) \quad \frac{\Delta' \varepsilon}{\varepsilon} \ll \frac{\lambda^{(i)}}{\Delta X_v}.$$

Nimmt man $\Delta X_v \approx 100 \lambda^{(i)}$ (um (6) zu erfüllen), so ist $\Delta \varepsilon / \varepsilon \ll 1/100$. Größenordnungsmäßig ist $\Delta' \varepsilon \sim \Delta \varepsilon \geq |\varepsilon_n - \varepsilon_m|$ die makroskopische Meßgenauigkeit (wie wir später sehen werden); daher ist (27), (29) gut erfüllbar und vernünftig.

3. - Dynamik.

3.1. *Die dynamischen makroskopischen Operatoren.* - Neben der Energie betrachten wir zunächst die Teilchendichte. Sie muß, da es sich um mehrere (N) Teilchen handelt, der Quantenfeldtheorie entnommen werden⁽¹⁴⁾. Das liefert in bekannter Weise für die mikroskopische Ladungsdichte an der Stelle \mathbf{r} in der Energiedarstellung

$$(30) \quad \varrho_{mn}(\mathbf{r}, t) = \int_V d\mathbf{r}_2 \dots \int_V d\mathbf{r}_N \psi_m^*(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) \psi_n(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N).$$

Dabei ist über das ganze Volumen V integriert; konsequenterweise sind ψ_n und ψ_m symmetrisch (oder antisymmetrisch) zu wählen. (Dieser Ausdruck läßt sich auch aus der gewöhnlichen Quantentheorie des N -Teilchen-Problems erraten). Dementsprechend werden wir als «makroskopische Dichte» die Größe

$$(31) \quad R_{mn}(\mathfrak{R}, t) = \frac{1}{\Delta V} \int_{\Delta V(\mathfrak{R})} d\mathbf{r}_1 \varrho_{mn}(\mathbf{r}_1, t) = \frac{1}{\Delta V} \int_{\Delta V(\mathfrak{R})} d\mathbf{r}_1 \int_V d\mathbf{r}_2 \dots d\mathbf{r}_N \psi_m^*(\mathbf{r}_1, \dots, \mathbf{r}_N) \psi_n(\mathbf{r}_1, \dots, \mathbf{r}_N)$$

eingeführen (Integration über das «kleinste makroskopische Volumen» $\Delta V(\mathfrak{R})$ an der Stelle \mathfrak{R}). Dann ist

$$(32) \quad \int d\mathfrak{R} R_{mn}(\mathfrak{R}, t) = \int_V d\mathbf{r}_1 \varrho_{mn}(\mathbf{r}_1, t) = N,$$

wie es sein muß, wenn ψ_n auf N normiert ist.

⁽¹⁴⁾ z.B. G. SÜSSMANN: *Einführung in die Quantenelektrodynamik* (Berlin, 1950); oder auch als klassische Feldtheorie: LUDWIG: *l.c.*, S. 19.

Daß es sich wirklich um eine makroskopische Observable handelt, ist leicht zu sehen: Die in (31) stehenden Integrale sind « beinahe » Normierungsintegrale (wenn $\Delta V(\mathfrak{R})$ genügend groß):

$$(33) \quad R_{mn}(\mathfrak{R}) = 0$$

für $|\varepsilon_m - \varepsilon_n|$ groß.

Das stimmt umso besser, je genauer die Bedingungen (6), (7) und (8) für ψ_n und ΔV erfüllt sind.

Die mikroskopische Stromdichte ist ebenfalls aus der Quantentheorie der Wellenfelder zu entnehmen:

$$(34) \quad j_{\nu}(\mathbf{r}, t) = \frac{i\hbar}{2m} \int_V d\mathbf{r}_2 \dots d\mathbf{r}_N \left\{ \frac{\partial}{\partial x_\nu} \psi_m^*(\mathbf{r}, \mathbf{r}_1, \dots, \mathbf{r}_N) \psi_n(\mathbf{r}, \mathbf{r}_1, \dots, \mathbf{r}_N) - \right. \\ \left. - \psi_m^*(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) \frac{\partial}{\partial x_\nu} \psi_n(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) \right\}.$$

(Ohne auf die Feldtheorie zurückzugreifen, kann man die Richtigkeit dieses Stromausdruckes durch Nachweis der Relation $\operatorname{div} \mathbf{j} = -\dot{\rho}$ unter Verwendung der Schrödingergleichung überprüfen). Der makroskopische Strom ist dann

$$(35) \quad J_{\nu}(\mathfrak{R}, t) = \frac{1}{\Delta V} \int_{\Delta V} j_{\nu}(\mathbf{r}, t) d\mathbf{r} = \\ = \frac{i\hbar}{2m \Delta V} \int_{\Delta V(\mathfrak{R})} d\mathbf{r}_1 \int_V d\mathbf{r}_2 \dots d\mathbf{r}_N \left\{ \psi_m^*|_V(\mathbf{r}_1, \dots, \mathbf{r}_N) \psi_n(\mathbf{r}_1, \dots, \mathbf{r}_N) - \right. \\ \left. - \psi_m^*(\mathbf{r}_1, \dots, \mathbf{r}_N) \psi_n|_V(\mathbf{r}_1, \dots, \mathbf{r}_N) \right\}.$$

Man überlegt wie oben, daß J_ν makroskopische Variable ist.

Alle anderen Größen wollen wir aus J_ν und R gewinnen. Die Form (13) (mit Integralen über $\varepsilon_m - \varepsilon_n = \hbar\omega$) wollen wir erst bei Bedarf benutzen.

Wir führen die folgenden Rechnungen für den Fall der Bosestatistik durch. Die Fermistatistik liefert im wesentlichen dieselben Ergebnisse. Wir setzen also an:

$$(36) \quad \psi_n = \frac{1}{N!} \sum_p A_n(1 \dots N) \exp \left[\frac{i}{\hbar} W_n(1 \dots N) - \frac{i}{\hbar} \varepsilon_n t \right]$$

und erhalten:

$$(37) \quad R_{mn}(\mathfrak{R}, t) = \frac{1}{N!^2} \sum_{pp'} A_{mn}^{pp'}(\mathfrak{R}, t)$$

mit der Abkürzung

$$(38) \quad A_{mn}^{pp'}(\mathfrak{R}, t) = \frac{1}{\Delta V} \int_{\Delta V(\mathfrak{R})} d\mathbf{r}_1 \int_{\Delta V} d\mathbf{r}_2 \dots d\mathbf{r}_N A_n(1 \dots N) A_m(1' \dots N') \cdot \exp \left[\frac{i}{\hbar} [W_n(1 \dots N) - W_m(1' \dots N')] \right].$$

Ferner:

$$J_{mn}^p(\mathfrak{R}, t) = \frac{i\hbar}{2m\Delta V} \exp \left[\frac{i}{\hbar} (\varepsilon_m - \varepsilon_n)t \right] \int_{\Delta V(\mathfrak{R})} d\mathbf{r}_1 \int_{\Delta V} d\mathbf{r}_2 \dots d\mathbf{r}_N \cdot \sum_{pp'} \left\{ (\log A_m(1' \dots N'))_{|_1} - (\log A_n(1 \dots N))_{|_1} - \frac{i}{\hbar} (W_n(1 \dots N) + W_m(1' \dots N'))_{|_1} \right\} \cdot A_n A'_m \exp \left[\frac{i}{\hbar} (W_n - W'_m) \right].$$

Berücksichtigen wir nun noch, daß $A_{m|_1}$ und $A_n A_{m|_1}$ usw. in $\Delta V(\mathfrak{R})$ bei unserer Näherung von \mathbf{r}_1 unabhängig sind, so wird

$$(39) \quad J_{mn}^p(\mathfrak{R}, t) = \frac{1}{N! 2m} \sum_{pp'} \left\{ i\hbar \left(\log \frac{A_m(\mathfrak{R}', \mathbf{r}_1, \dots, \mathbf{r}'_N)}{A_n(\mathfrak{R}, \mathbf{r}_2, \dots, \mathbf{r}_N)} \right)_{|_1}^{\frac{pp'}{mn} \mathfrak{R}t} + (W'_m + W_n)_{|_1}^{\frac{pp'}{mn} \mathfrak{R}t} \right\},$$

mit der Abkürzung

$$(40) \quad \overline{f_{mn}(\mathbf{r}, \mathbf{r}')^{\frac{pp'}{mn} \mathfrak{R}t}} = \frac{1}{\Delta V} \int_{\Delta V(\mathfrak{R})} d\mathbf{r}_1 \int_{\Delta V} d\mathbf{r}_2 \dots d\mathbf{r}_N f_{mn}(\mathbf{r}_1, \dots, \mathbf{r}_N; \mathbf{r}'_1 \dots \mathbf{r}'_N) \cdot A_n A'_m \exp \left[\frac{i}{\hbar} (W_n - W'_m) \right].$$

Ferner wird (unter Verwendung unserer Näherungsannahmen (6), (7), (8))

$$(41) \quad J_{mn}^{\nu\mu}(\mathfrak{R}, t) = \frac{1}{2mN! 2} \sum_{pp'} \left\{ \left(\log \frac{A'_m(\mathfrak{R}, \dots)}{A_n(\mathfrak{R}, \dots)} \right)_{|_1} (W_n - W'_m)_{|_1} + \overline{(W'_m + W_n)_{|_1} (\mathfrak{R}, \dots) + \log A'_m(\mathfrak{R}, \dots) \cdot \log A_n(\mathfrak{R}, \dots) (W'_m + W_n)_{|_1}} + \frac{i}{\hbar} \overline{(W'_m + W_n)_{|_1} (W_n - W'_m)_{|_1}} \right\}.$$

3.2. Kontinuitätsgleichung. – Nach diesen Vorbereitungen ist die Ableitung der Kontinuitätsgleichung (und damit die Bestätigung, daß J_ν zur Dichte R

passend definiert ist) sehr einfach. Sie gilt bereits für jedes Matricelement:

$$(42) \quad \boxed{\dot{R}_{mn}(\mathfrak{R}, t) = -J_{\nu\mu}{}_{mn}(\mathfrak{R}, t)}.$$

Zunächst ist (nach (4a) mit (3a))

$$(43) \quad -\dot{R}_{mn}(\mathfrak{R}, t) = \frac{i}{\hbar}(\varepsilon_n - \varepsilon_m)R_{mn}(\mathfrak{R}, t) = \frac{i}{N!2m\hbar} \sum_{i=1}^N \{ (W_{n|v})^2 - (W_{m|v})^2 \} R_{mn},$$

wobei man in $\sum_i \{ (W_{n|v})^2 - (W_{m|v})^2 \}$ beliebige Argumente (in den beiden Gliedern in beliebig verschiedenen Permutationen) einsetzen kann. Daher können wir $\sum_i \{ (W_{n|v})^2 - (W'_{m|v})^2 \}$ in das in R_{mn} stehende Integral passend hineinziehen und nur $r_1 = \mathfrak{R}$ setzen. Dann wird also

$$(44) \quad -\dot{R}_{mn}(\mathfrak{R}, t) = \frac{i}{N!2m\hbar} \sum_{i=1}^N \sum_{\nu\nu'} \{ (W_{n|v})^2 - (W_{m|v})^2 \}^{\nu\nu' mn \mathfrak{R}t}.$$

Betrachten wir nun $J_{\nu\mu}{}_{\nu}$ nach (41): Wir ersetzen das Symbol $\partial/\partial x_\nu(1)$ (d.h. $|_1\nu$) durch $\partial/\partial x_\nu(i)$ (d.h. $|_i\nu$) und summieren über i von 1 bis N . Durch partielle Integration kann man dann leicht zeigen, daß alle Glieder mit $i \neq 1$ sich gegenseitig kompensieren, sodaß man also nichts hinzugefügt hat⁽¹⁵⁾. Dann aber ist (44) mit dem letzten Glied von $J_{\nu\mu}{}_{\nu}$ nach (41) identisch, und alle übrigen Glieder kompensieren sich wegen der Gleichung (46).

Wir haben bei der Herleitung von $J_{\nu\mu}{}_{\mu}$ von der Beziehung (26) Gebrauch gemacht, also (s. S. 1068) von der Abschätzung

$$\frac{\Delta\varepsilon}{\varepsilon} = \frac{|\varepsilon_n - \varepsilon_m|}{(\varepsilon_n + \varepsilon_m)/2} \ll \frac{\lambda}{\Delta X_\nu}.$$

Dadurch ist — wie bereits früher bemerkt — die *makroskopische* Energiemeßgenauigkeit mit der *makroskopischen* Ortsmeßgenauigkeit in Beziehung gesetzt, weil $\Delta\varepsilon$ nach 1·2 wirklich die Energieunschärfe bedeutet.

3·3. Bewegungsgleichung. — Wegen des bekannten Zusammenhangs zwischen der makroskopischen Geschwindigkeit V_ν eines herausgegriffenen Volumenelementes mit der Stromdichte und Dichte

$$(45) \quad V_\nu = \frac{J_\nu}{R}$$

⁽¹⁵⁾ Man sieht dies sofort, wenn man nach (35) $J_{\nu|v}$ bildet; dann kann man offenbar $\partial^2/\partial x_{\nu(1)}^2$ durch $\sum_{i=1}^N \partial^2/\partial x_{\nu(i)}^2$ ersetzen, ohne etwas zu ändern (zweimalige partielle Integration).

ist auch die Gewinnung dieser Größe aus der Quantentheorie vorgeschrieben. Es treten jedoch jetzt Produkte von Observablen auf, sodaß man nicht immer mit der Betrachtung von Matrixelementen auskommen wird.

Uns interessiert die « Kraftdichte »

$$(46) \quad R \frac{dV_v}{dt} \equiv R \left\{ \frac{\partial V_v}{\partial t} + V_\mu V_{v|\mu} \right\} = \dot{J}_v + \left(\frac{J_v J_\mu}{R} \right)_{|\mu}$$

(unter Verwendung von (45) und der Kontinuitätsgleichung). Dabei ist benutzt, daß die Reihenfolge makroskopischer Observablen beliebig geändert werden darf.

Wir wollen zeigen, daß sich auch \dot{J}_v als Tensordivergenz schreiben läßt. Um dies einzusehen, lassen wir uns von dem bekannten Ausdruck für den Spannungstensor der Elektronenfeldtheorie⁽¹⁴⁾ leiten, der (für ein klassisches Einteilchenfeld ohne Wechselwirkung mit sich selbst)

$$\tau_{v\mu} = \frac{\hbar^3}{2m^2} \left\{ \psi_v^* \psi_{|\mu} + \psi_{|\mu}^* \psi_v - \frac{\delta_{v\mu}}{2} (\psi^* \psi)_{|\lambda|\lambda} \right\}$$

lautet; er ist so bestimmt, daß für die Stromdichte j_v gilt:

$$\frac{\partial}{\partial t} j_v + \tau_{v\mu|\mu} = V_v^{(a)} \cdot \varrho,$$

($V^{(a)}$ = äußeres Feld, $\varrho = \psi^* \psi$ = Dichte).

In Analogie zu unseren früheren Ableitungen wird man als Spannungstensor

$$(47) \quad \tau'_{v\mu} = \frac{\hbar^3}{2m^2 \Delta V} \int d\mathbf{r}_1 \int d\mathbf{r}_2 \dots d\mathbf{r}_n \left\{ \psi_{m1}^* \psi_{n|\mu} + \psi_{m|\mu}^* \psi_{n1} - \frac{1}{2} \delta_{v\mu} (\psi_m^* \psi_n)_{11|\lambda|\lambda} \right\}$$

vermuten. Dabei haben wir wieder die Symmetrie der Wellenfunktion benutzt. Offenbar ist $\tau'_{v\mu}$ makroskopische Observable. Setzen wir nun den Ansatz (36) ein, so wird:

$$(48) \quad \tau'_{v\mu}(\mathfrak{R}, t) = \frac{i\hbar^2}{2m^2} \sum_{\nu\nu'} \left\{ \overline{(\log A'_m A_n)_{1\nu}} (W_n + \overline{W'_m})_{1\mu} + \overline{(\log A'_m A_n)_{1\mu}} (W_n + \overline{W'_m})_{1\nu} - \right. \\ \left. - \frac{i}{\hbar} \overline{W'_{m1\nu}} W_{n1\mu} + \overline{W_{m1\mu}} W'_{n1\nu} + \delta_{v\mu} \left[2(\log A'_m A_n)_{1\lambda} (\overline{W_n} - \overline{W'_m})_{1\lambda} + \right. \right. \\ \left. \left. + (\overline{W_n} - \overline{W'_m})_{1\lambda|\lambda} + \frac{i}{\hbar} (\overline{W_{n1\lambda}} - \overline{W'_{m1\lambda}})^2 \right] \right\}.$$

Eine etwas umständliche aber elementare Rechnung ergibt dann (wenn man an geeigneter Stelle ähnlich wie bei der Kontinuitätsgleichung $\partial/\partial x_\nu(1)$ durch $\sum_{i=1}^N \partial/\partial x_\nu(i)$ ersetzt, was auch hier kleine Abänderung bedeutet) das gewünschte Ergebnis:

$$\tau'_{\nu\mu|\mu}(\mathfrak{R}, t) = -\frac{\partial}{\partial t} J_{mn}(\mathfrak{R}, t) + \frac{\hbar^3}{2m^2} \sum_{\nu\mu} \overline{V_1}_\nu.$$

In unserer Näherung läßt sich auch $V_1|_\nu = \sum_{i \leq k} V_{ik}|_\nu$ als Tensordivergenz schreiben: Hängt ein V_{ik} nur von $|\mathbf{r}_i - \mathbf{r}_k| = |\mathbf{r}^{(ik)}|$ ab, so können wir schreiben:

$$\overline{V_1}|_\nu = \sum_{i \leq k} \int_{\Delta V(\mathfrak{R})} d\mathbf{r}_1 \int_{\mathfrak{V}} d\mathbf{r}_2 \dots d\mathbf{r}_n \frac{V'(|\mathbf{r}^{(1k)}|)}{|\mathbf{r}^{(1k)}|} \mathbf{r}_\nu^{(1k)} A_n A'_m \exp \left[\frac{i}{\hbar} (W_n - W'_m) \right].$$

Nach (4a) und (7) ist V' dabei als klein von 1. Ordnung anzusehen; die Reichweite soll höchstens von der Größenordnung ΔX_ν werden. Alle anderen Größen brauchen wir daher nur in nullter Näherung zu berücksichtigen. Wir betrachten eine bestimmte Permutation PP' und ein festes k ; als neue Variable führen wir statt $\mathbf{r}_k \mathbf{r}^{(1k)}$ ein. Dann wird (in unserer Näherung)

$$A_n(\mathbf{r}_1, \dots, \mathbf{r}_k, \dots) \approx A_n(\mathbf{r}_1, \dots, \mathbf{r}_1, \dots),$$

$$W_n(\mathbf{r}_1, \dots, \mathbf{r}_k, \dots) \approx W_n(\dots \mathbf{r}_1, \dots, \mathbf{r}_1) - (W_{n|\lambda}(\mathbf{r}_1, \dots \mathbf{r}_1 \dots) + W_{n|\lambda}(\mathbf{r}_1 \dots \mathbf{r}_1 \dots)) \mathbf{r}_\lambda^{(1k)}$$

(für m analog) und es ist sofort klar, daß man

$$(49) \quad \overline{V_1}|_\nu = \sum_{i \leq k} \frac{\Delta}{\Delta X_\mu} \int_{\Delta V(\mathfrak{R})} d\mathbf{r}_1 \int_{\mathfrak{V}} d\mathbf{r}_2 \dots \int_{\mathfrak{V}} d\mathbf{r}^{(1k)} \dots \int_{\mathfrak{V}} d\mathbf{r}_n \cdot$$

$$\frac{V'(|\mathbf{r}^{(1k)}|) \mathbf{r}_\nu^{(1k)} \mathbf{r}_\mu^{(1k)}}{|\mathbf{r}^{(1k)}| (i/\hbar) (W_{nm|\lambda}(\mathbf{r}_1 \dots \mathbf{r}_1 \dots) + W_{nm|\lambda}(\mathbf{r}_1 \dots \mathbf{r}_1 \dots)) \mathbf{r}_\lambda^{(1k)}} \cdot$$

$$A_n(\mathfrak{R} \dots \mathfrak{R} \dots) A_m(\mathfrak{R} \dots \mathfrak{R} \dots) \exp \left[\frac{i}{\hbar} W_{nm}(\mathbf{r}_1 \dots \mathbf{r}_1 \dots) \right]$$

setzen kann. Als Resultat unserer Überlegungen erhalten wir so die «hydro-

dynamische Grundgleichung »

$$(50) \quad R \left\{ \frac{\partial V_v}{\partial t} + V_\mu \dot{V}_{v\mu} \right\}_{\mathfrak{R}, t} = T_{v\mu\mu}(\mathfrak{R}, t) + V_{\mu}^{(\omega)} \cdot R$$

mit

$$(51) \quad T_{v\mu}(\mathfrak{R}, t) = \frac{1}{\sigma(\varepsilon)} \int \varrho(\varepsilon, \omega) \left\{ \left(\frac{J_v J_\mu}{R} \right)_{\varepsilon, \omega} - (\tau'_{v\mu})_{\varepsilon, \omega} + (V_{v\mu})_{\varepsilon, \omega} \right\}_{\mathfrak{R}t} \exp[i\omega(t - \tau)] d\omega$$

wobei J_μ durch (35) bzw. (39), $\tau'_{v\mu}$ durch (47) bzw (48) und $V_{v\mu}$ durch

$$(52) \quad V_{v\mu}(\mathfrak{R}, t) = \frac{\hbar^2}{2m^2} \sum_{k \geq 1} \sum_{\mathfrak{P}\mathfrak{P}'} \int_{\Delta V(\mathfrak{R})} d\mathbf{r}_1 \int_V d\mathbf{r}_2 \dots d\mathbf{r}_n \cdot$$

$$\cdot \frac{V'(|\mathbf{r}^{(1k)}|) r_v^{(1k)} r_\mu^{(1k)}}{|\mathbf{r}_k^{(1k)}| [W_{nm|1}(\mathbf{r}_1, \dots, \mathbf{r}_1, \dots) + W_{nm|k}(\mathbf{r}_1 \dots \mathbf{r}_1 \dots)] r_\lambda^{(1k)}} \cdot$$

$$\cdot A_n(\mathfrak{R}, \dots, \mathfrak{R}, \dots) A'_m(\mathfrak{R}, \dots, \mathfrak{R}, \dots) \exp \left[\frac{i}{\hbar} W_{nm}(\mathbf{r}_1, \dots, \mathbf{r}_1, \dots) \right]$$

definiert ist; $V^{(\omega)}$ ist ein äußeres Feld. $V_{v\mu}$ ist nur dann makroskopische Observable, wenn die Voraussetzungen (6) (7) (8) über φ_n gemacht werden.

Wir haben alle Überlegungen für die Bosestatistik durchgeführt. Für die Fermistatistik ändert sich an Grundsätzlichem nichts, nur muß in allen Größen $\sum_{\mathfrak{P}\mathfrak{P}'}$ durch $\sum_{\mathfrak{P}\mathfrak{P}'} (-1)^{\mathfrak{P}+\mathfrak{P}'}$ ersetzt werden. Obwohl die Thermodynamik erst später behandelt werden soll, sei hier schon bemerkt, daß die Theorie natürlich auch für den Fall eines statistischen Gemisches gilt, sofern $w(\varepsilon, \omega)$ nicht zu unvernünftig ist (es darf nicht die charakteristischen Eigenschaften der m.O. zerstören).

4. — [Diskussion der Ergebnisse.

4.1. *Vergleich mit der klassischen Theorie.* — Wir haben das gewünschte Ergebnis erhalten, daß die klassische makroskopische (phänomenologische) Theorie eine Folge der Quantentheorie ist. Man kann natürlich nicht erwarten, daß eine klassische *mikroskopische* Theorie mit unserer in allen Zügen übereinstimmt: Die eingehenden Größen werden zwar in derselben Beziehung zueinander stehen, jedoch durch die grundsätzlichen großen Unterschiede « im Kleinen » verschieden strukturiert sein.

Eine vollständige klassische Beschreibung findet sich z.B. bei GREEN ⁽¹⁶⁾. Wir entnehmen für den Spannungstensor (nach passender Umformung) die Größe

$$(53) \quad T_{\nu\mu}^{\text{Klass.}}(\mathfrak{R}, t) = \frac{j_\nu j_\mu}{\varrho} - \int f(\mathbf{v}, \mathfrak{R}, t) v_\nu v_\mu d\mathbf{v} + \frac{1}{2} \int n_2(\mathfrak{R}, t; \mathbf{r}) \frac{V'(\mathbf{r})}{r} r_\nu r_\mu d\mathbf{r}.$$

Dabei ist $f(\mathbf{v}, \mathfrak{R}, t)$ die Geschwindigkeitsverteilung der Moleküle an der Stelle \mathfrak{R} (Zahl der Moleküle je cm^3 und $(\text{cm/s})^3$), $n_2(\mathfrak{R}, t; \mathbf{r})$ die Wahrscheinlichkeitsverteilung für das Auffinden eines zweiten Moleküls im Abstand r , wenn das erste sich an der Stelle \mathfrak{R} befindet. Die Korrespondenz des ersten und letzten Gliedes zu den entsprechenden in (51) ist offensichtlich. Das erste gibt den Beitrag der reinen Konvektion; das letzte die unmittelbare Potentialwechselwirkung der Moleküle. Das mittlere Glied vergleicht man zweckmäßig mit $\tau'_{\nu\mu}$ nach (47); es stellt offenbar den dissipativen Anteil der ungeordneten Bewegung dar; die Korrespondenz zur Quantentheorie ist hier nicht ganz so deutlich; immerhin enthält letztere auch eine Art Mittelwerte über den Tensor $v_\nu v_\mu$; hinzu kommt noch ein Diagonalglied, das man als kinetische Energie deuten kann (vielleicht würde eine strengere klassische Theorie auch noch solche Diagonalglieder liefern; diese Frage soll hier nicht weiter verfolgt werden).

Zusammenfassend kann also gesagt werden, daß auch gewisse mikrophysikalische Korrespondenzen erhalten bleiben.

4.2. *Vergleich mit der Entwicklung nach \hbar .* — Wir beziehen uns auf die Darstellung von GREEN ⁽¹⁷⁾, bemerken aber, daß zahlreiche Autoren dasselbe Verfahren anwenden. (Es geht letztlich auf WIGNER ⁽¹⁸⁾ zurück).

Die durch

$$(54) \quad (x|W_i|x') \equiv W(\mathbf{r}_i, \mathbf{r}'_i, t) = \int f(\mathbf{v}_i, \mathfrak{R}_i, t) \exp \left[\frac{i}{\hbar} m \sum_i \mathbf{v}_i \mathbf{r}_i \right] d\mathbf{v}_1 \dots d\mathbf{v}_N$$

definierten Fourierkomponenten der Dichtematrix W ($\mathfrak{R}_i = (\mathbf{r}_i + \mathbf{r}'_i)/2$) kann als eine Art Geschwindigkeitsverteilungsfunktion angesehen werden. Bildet man nämlich (in geeigneter Bezeichnungsweise) mit $(x|\mathbf{v}_i|x') = (\hbar/im)(d/dx) \cdot \delta(x - x')$ als Matricelement des Geschwindigkeitsoperators in der Ortsdarstellung das Produkt $\mathbf{v}_i \cdot W$, so wird

$$(55) \quad (x|\mathbf{v}_i \cdot W|x) = \int d\mathbf{x} (x|\mathbf{v}_i|x')(x'|W|x) = \int f(\mathbf{v}_1, \mathfrak{R}, t) \mathbf{v}_i d\mathbf{v}_1 \dots d\mathbf{v}_N.$$

⁽¹⁶⁾ H. S. GREEN: *l.c.*, S. 132.

⁽¹⁷⁾ H. S. GREEN: *l.c.*, S. 235.

⁽¹⁸⁾ E. WIGNER: *Phys. Rev.*, **40**, 749 (1932).

Das entspricht der klassischen mittleren Geschwindigkeit eines einzelnen Teilchens an der Stelle \mathfrak{R} ; die richtige mittlere Geschwindigkeit ist nach den Prinzipien der Quantentheorie jedoch

$$(56) \quad \text{Sp } (W v_i) = \int dx \int dx' (x | v_i | x') (x' | W | x) = \int d\mathfrak{R} \int f(v_i, \mathfrak{R}, t) v_i dv_1 \dots dv_N,$$

also etwas ganz anderes als der obige Ausdruck (55). Das ist natürlich eine Folge der Unschärfebeziehung, die ja verbietet, von Ort und Geschwindigkeit eines Teilchens zu reden. Es ist diese Größe f also in Strenge keinesfalls eine Geschwindigkeitsverteilungsfunktion.

Wir haben oben ausdrücklich von einem *einzelnen* Teilchen gesprochen: Das bringt zum Ausdruck, daß dieses Verfahren niemals zur Gewinnung der statistischen Mechanik verwendet werden kann. Hierauf hat VAN KAMPEN ⁽⁶⁾ (hinsichtlich der Wigner'schen Verteilungsfunktion) bereits hingewiesen.

Obwohl wir es demnach ablehnen müssen, f als physikalisch sehr bedeutende Größe zu betrachten, wollen wir doch noch auf das Verfahren zur Gewinnung des klassischen Grenzfalles eingehen. W erfüllt wegen der bekannten Bewegungsgleichung von W die Gleichung

$$(57) \quad -\frac{\partial f(v, \mathfrak{R}, t)}{\partial t} = \sum_i v_i \frac{\partial f(v, \mathfrak{R}, t)}{\partial \mathfrak{R}_i} + \frac{i}{\hbar} \left(\frac{m}{2\pi\hbar} \right)^{3N} \int dv_1^0 \dots dv_N^0 \int dr_1 \dots dr_N \cdot \\ \cdot \left\{ V\left(\mathfrak{R}_i - \frac{r_i}{2}\right) + V\left(\mathfrak{R}_i + \frac{r_i}{2}\right) \right\} f(v_i^0, \mathfrak{R}, t) \exp \left[\frac{i}{\hbar} m \sum_i (v_i^0 - v_i) r_i \right].$$

Entwickelt man nun $f(v_0)$ in eine Taylorreihe um die Stelle r , V in eine solche an der Stelle \mathfrak{R} , so erhält man

$$(58) \quad -\partial f / \partial t \approx \sum_i \left(v_i \frac{\partial f}{\partial \mathfrak{R}_i} - \frac{1}{2m} \frac{\partial V}{\partial \mathfrak{R}_i} \cdot \frac{\partial f}{\partial v_i} \right) + \frac{1}{3!} \frac{(\frac{1}{2}\hbar)^2}{m^3} \sum_{i,k,j} \frac{\partial^3 V}{\partial \mathfrak{R}_i \partial \mathfrak{R}_k \partial \mathfrak{R}_j} \cdot \\ \cdot \frac{\partial^3 f}{\partial v_i \partial v_j \partial v_k} + \hbar^4 \dots$$

Für $\hbar = 0$ folgt die Bewegungsgleichung der klassischen Verteilungsfunktion. Nach unseren Überlegungen sollte man statt des Grenzübergangs $\hbar \rightarrow 0$ lieber $\partial^3 V / \partial \mathfrak{R}_i \partial \mathfrak{R}_k \partial \mathfrak{R}_j$ und $\partial^3 f / \partial v_i \partial v_j \partial v_k$ als klein ansehen: die damit an die Zustände φ_n und an V gestellten Forderungen entsprechen in jeder Hinsicht den von uns in dieser Arbeit erhobenen, wie man sich leicht überlegt. Was wir eigentlich gewonnen haben ist also weniger das Verständnis des Überganges $\hbar \rightarrow 0$ als den Einbau der Konzeption einer makroskopischen Messung, indem wir statt einer — quantentheoretisch bedeutungslosen — Verteilungsfunktion direkt die Observablen untersucht haben.

Herrn Prof. LUDWIG habe ich für anregende Diskussionen sehr zu danken.

RIASSUNTO (*)

Il passaggio alla Fisica classica per mezzo della condizione limite $\hbar \rightarrow 0$ non è sempre giustificato. Lo si sostituisce pertanto con opportune premesse sugli stati e gli autovalori di un sistema di particelle. Si considerano come macroscopicamente ammissibili solo quelle osservabili che soddisfano le condizioni formulate da LUDWIG ⁽¹⁾. Basandosi sulla dinamica dei gas si mostra che la Fisica classica è una conseguenza della teoria quantica, in quanto si introducono le osservabili macroscopiche densità, densità di corrente e tensore degli sforzi e se ne esaminano le equazioni del moto. In un successivo lavoro si tratterà della Termodinamica.

(*) Traduzione a cura della Redazione.

General Theory of Propagators.

H. UMEZAWA (*) and A. VISCONTI (+)

The Physical Laboratories - University of Manchester

(ricevuto il 15 Marzo 1955)

Summary. — A general theory of propagators is developed. All propagators are derived from a generating functional. Renormalization corresponds to the adjustment of the coupling constants of the external sources. As an example, renormalized equations for Compton and Møller scattering are derived.

1. — Introduction.

SCHWINGER ⁽¹⁾ has proposed a theory of propagators ^(§) which has several patent advantages over previous ones based on perturbation theory. The purposes of the present paper ^(×) are: 1) to give a general scheme for this theory by starting from a new representation (source-free representation) which permits one to obtain all propagators and their equations from a generating functional and a set of basic equations, 2) to derive new equations, different from those of Bethe-Salpeter, for Compton and Møller scattering. These equations make the renormalisation procedure automatic and they can be treated by a covariant procedure in which successively increasing numbers of virtual photons and pairs are taken into account. It seems interesting to

(*) On leave from Nagoya University, Japan.

(+) On leave from the Institut Henri Poincaré, Paris; now with CERN (European Organization for Nuclear Research), Theoretical Study Division at the Institute for Theoretical Physics, Copenhagen.

(§) Propagators are sometimes called GREEN functions. see, e.g., J. SCHWINGER ⁽¹⁾.

(×) An abstract of the present theory has been given in a series of publications in *Comptes Rendus* ⁽²⁾.

(1) J. SCHWINGER: *Proc. Nat. Acad.*, **37**, 452, 455 (1951).

(2) H. UMEZAWA and A. VISCONTI: *Compt. Rend. (Paris)*, **239**, 690, 749, 1466 (1954).

note that the solutions of both equations can be obtained by means of only one equation, namely for their common resolvent.

The paper is divided into four parts: the first part, sections 2, 3, and 4, contains a concise formulation of the theory of propagators for a system of particles. Following SCHWINGER, we introduce external sources; but in addition we choose a representation in which the electron and electromagnetic fields obey the same equations as they would in Heisenberg representation without sources. The sources influence therefore the state vector only, not the field operators. This representation is useful because it enables us to define initial and final states as eigenstates of the Hamiltonian without sources. The time variation of the state vector is given by a unitary evolution operator U which is used as generating functional for all many-body propagators. From the equations U has to obey, the functional equations for the propagators are derived.

In the second part, which comprises sections 4 and 5, it is shown that the many-body-propagators give physical information about the system (theorems I and II of section 4) and have an intuitive graphical interpretation.

Sections 6, 7 and 8 (the third part) deal, as an example, with the two-body-propagators. The connection between the present theory and the Bethe-Salpeter equation is studied. The integral equations found for Møller and Compton scattering differ from the Bethe-Salpeter equation and have the important property of «renormalisation invariance». The solution of these integral equations can be represented in terms of «skeletons» (*), i.e. irreducible diagrams which include neither self-energy nor vertex diagrams (+). In section 8, the external lines of these irreducible diagrams are shown to correspond to the «free dressed» particle: the particle with the experimental mass and charge.

In sections 9 and 10, we study the connection between the present theory and the conventional renormalisation. We shall show that our theory can be made to agree with the renormalisation theory.

The last part, section 11, is devoted to the study of the solutions of our equations for Møller and Compton scattering.

(*) A provisional approach to this problem has been given by S. TANAKA and H. UMEZAWA (3).

(+) It has been pointed out by H. S. GREEN (4) that elimination of all uncorrected quantities γ_μ , S_F , D_F , by means of quantities including radiative corrections might lead to a renormalized theory. This suggestion is followed up in sections 9 and 10.

(3) S. TANAKA and H. UMEZAWA: *Prog. Theor. Phys.*, **10**, 617 (1953).

(4) H. S. GREEN: *Proc. Phys. Soc.*, A **66**, 873 (1953).

2. - Formulation of the Theory.

Let us consider an electron-positron field $\psi(x)$, $\bar{\psi}(x)$ interacting with an electromagnetic field $A_\mu(x)$.

The external sources of these fields we will denote by η , $\bar{\eta}$, and J_μ respectively. In the representation chosen by us, which we shall call the source-free representation, the field equations are

$$(2.1a) \quad \bar{F}(x)\psi(x) \equiv (\gamma_\mu \partial_\mu + m - ie\gamma_\mu A_\mu(x))\psi(x) = 0$$

$$(2.1b) \quad \bar{\psi}(x)\bar{F}(x) \equiv \bar{\psi}(x)(\gamma_\mu \partial_\mu + m + ie\gamma_\mu A_\mu(x)) = 0$$

$$(2.1c) \quad \square A_\mu(x) = -j_\mu(x) \equiv -\frac{e}{2} \bar{\psi}(x)\gamma_\mu\psi(x) + \text{h. c.}$$

If we denote by ψ' , $\bar{\psi}'$, A'_μ the same fields in Heisenberg representation their field equations are given by

$$(2.2a) \quad \bar{F}(x)\psi'(x) = \lambda\eta(x)$$

$$(2.2b) \quad \bar{\psi}'(x)\bar{F}(x) = \lambda\bar{\eta}(x)$$

$$(2.2c) \quad \square A'_\mu(x) = -j_\mu(x) - \lambda' J_\mu(x),$$

where λ and λ' are the coupling constants of the external sources. This corresponds with an interaction Hamiltonian due to the sources

$$(2.3) \quad \mathcal{H}'_s(x) = -\lambda\bar{\psi}'(x)\eta(x) - \lambda\bar{\eta}(x)\psi'(x) - \lambda' J_\mu(x) A'_\mu(x).$$

It is shown in section 3 that all meaningful results are obtained by taking $\eta = \bar{\eta} = J_\mu = 0$ at the end of the calculations. The system then becomes conservative.

Let us assume that the state vectors Ψ , Ψ' in these two representations are connected by the unitary transformation U and that Ψ and Ψ' agree on a surface σ_{II} . Then we have

$$(2.4) \quad \Psi[\sigma] = U[\sigma, \sigma_{\text{II}}, \eta, J]\Psi[\sigma_{\text{II}}],$$

(5) H. UMEZAWA and S. ONEDA: *Nuovo Cimento*, **12**, 566 (1954); H. UMEZAWA, J. PODOLANSKI and S. ONEDA: *Proc. Phys. Soc.*, A (in the press); T. KINOSHITA: *Phys. Rev.*, **96**, 199 (1954).

where the evolution operator U satisfies

$$(2.5) \quad i \frac{\delta}{\delta \sigma(x)} U[\sigma, \sigma_{II} \eta, J] = U[\sigma, \sigma_{II}, \eta, J] \mathcal{H}'_s(x).$$

As shown by UMEZAWA *et al.* ⁽⁵⁾, each term in the interaction Hamiltonian density must include an even number of operators, which anticommute with any given field operator. By taking $\psi'(x)$, $A'_\mu(x)$, $\eta(x)$ and $J_\mu(x)$ successively as test operators, we deduce from (2.3)

$$(2.4a) \quad [\eta(x), \bar{\psi}'(x')]_+ = [\eta(x), \psi'(x')]_+ = [\eta(x), \eta(x')]_+ = 0$$

$$(2.4b) \quad [\eta(x), A'_\mu(x')]_- = [\eta(x), J_\mu(x')]_- = 0$$

$$(2.4c) \quad [J_\mu(x), \psi'(x')]_- = [J_\mu(x), \bar{\psi}'(x')]_- = 0$$

$$(2.4d) \quad [J_\mu(x), A'_\mu(x')]_- = [J_\mu(x), J_\mu(x')]_- = 0.$$

These commutation relations are proved to be valid only when x and x' have a space-like distance. We can, however, assume them to obtain for any two points x , x' without introducing a contradiction to the field equations. This implies that the operators η , $\bar{\eta}$, J_μ are quite independent of the electron-photon system, since they commute with the Hamiltonian at arbitrary points; there is no reaction of the field on the sources.

Denoting by H the Hamiltonian in the source-free representation, its eigenstates Ψ_n are defined by

$$(2.7) \quad H\Psi_n = E_n\Psi_n.$$

It is an advantage of the present representation that the states Ψ_n are independent of the sources. The vacuum Ψ_0 is defined as the state of lowest energy E_0 . Then

$$(2.8a) \quad H\psi^\pm(x, E)\Psi_n = (\mp E + E_n)\psi^\pm(x, E)\Psi_n$$

$$(2.8b) \quad HA^\pm_\mu(x, E)\Psi_n = (\mp E + E_n)A^\pm_\mu(x, E)\Psi_n$$

$$(2.9) \quad \psi^+(x)\Psi_0 = \bar{\psi}^+(x)\Psi_0 = A^+_\mu(x)\Psi_0 = 0,$$

where $\psi^+(x, E)$, $A^+_\mu(x, E)$, $\psi^-(x, E)$, $A^-_\mu(x, E)$ are the positive and negative frequency parts of the fields. The interpretation of these operators as anni-

hilation and creation operators is obtained in a similar way to the free field case. Equations (2.8a) and (2.8b) show that $\psi^\pm(x, E)\Psi_n$, $\psi^\pm(x, E)\Psi_n$, $A_\mu^\pm(x, E)\Psi_n$ are eigenstates of the source-free Hamiltonian H ; therefore we shall define a state of n electrons, m positrons, and l photons by the state vector:

$$(2.10) \quad \psi^-(x_1) \dots \psi^-(x_n) \bar{\psi}^-(x'_1) \dots \bar{\psi}^-(x'_m) A_{\mu_1}^-(z_1) \dots A_{\mu_l}^-(z_l) \Psi_0.$$

3. - Definition of Propagators.

The equation (2.7) can be replaced by the integral equation

$$(3.1) \quad U[\sigma_I, \sigma_{II}, \eta, J] = 1 - i \int_{\sigma_{II}}^{\sigma_I} \mathcal{H}_s(x') U[\sigma', \sigma_{II}, \eta, J] d^4x',$$

where σ' passes through the point x' . We change \mathcal{H}_s by an infinitesimal amount $\delta\mathcal{H}$ and ask for the corresponding change δU .

When U and U_1 satisfy the operator equations:

$$U = 1 + (K_1 + K_2)U$$

$$U_1 = 1 + K_1 U_1,$$

we can prove ⁽⁶⁾:

$$U - U_1 = U_1 K_2 U.$$

From this follows, neglecting higher order terms,

$$(3.2) \quad \delta U[\sigma_I, \sigma_{II}, \eta, J] = -i \int_{\sigma_{II}}^{\sigma_I} U[\sigma_I, \sigma', \eta, J] \delta\mathcal{H}(x') U[\sigma', \sigma_{II}, \eta, J] d^4x'.$$

We now define $\delta\mathcal{H}$ as a change induced by a variation of the sources:

$$(3.3) \quad \delta\mathcal{H}(x) = -\lambda(\bar{\psi}(x)\delta\eta(x) + \delta\bar{\eta}(x)\psi(x)) - \lambda' \delta J_\mu(x) A_\mu(x),$$

where $\delta\eta$, $\delta\bar{\eta}$, δJ_μ have the same commutation properties as η , $\bar{\eta}$, J_μ .

⁽⁶⁾ A. VISCONTI: *Thèses (Paris)*, p. 75 (1953); *Journ. Phys. et Rad.*, **14**, 1951 (1953); *Compt. Rend. (Paris)*, **236**, 2489 (1953).

We define now the functional G, \mathcal{G}_μ by (*)

$$(3.4a) \quad G[\eta, J; x'] \equiv \frac{1}{i} \frac{\delta}{\delta \eta(x')} U[\sigma_I, \sigma_{II}, \eta, J]$$

$$(3.4b) \quad G[\eta, J; x] \equiv \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x)} U[\sigma_I, \sigma_{II}, \eta, J]$$

$$(3.4c) \quad \mathcal{G}_\mu[\eta, J; z] \equiv \frac{1}{i} \frac{\delta}{\delta J_\mu(z)} U[\sigma_I, \sigma_{II}, \eta, J].$$

From (3.3) we find

$$(3.5a) \quad G[\eta, J; x'] = \lambda(\sigma_I, \bar{\psi}(x'), \sigma_{II})$$

$$(3.5b) \quad G[\eta, J; x] = \lambda(\sigma_I, \psi(x), \sigma_{II})$$

$$(3.5c) \quad \mathcal{G}_\mu[\eta, J; z] = \lambda'(\sigma_I, A_\mu(z), \sigma_{II}),$$

where

$$(\sigma_I, A(x)B(x'), \sigma_{II}) \equiv U[\sigma_I, \sigma, \eta, J] A(x) U[\sigma, \sigma', \eta, J] B(x') U[\sigma', \sigma_{II}, \eta, J]$$

for any operators $A(x), B(x')$. We generalise the definition (3.4) to many-particle operators:

$$(3.6a) \quad G[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m] \equiv \\ \equiv (i)^{-n-m} \frac{\delta^{n+m}}{\delta \bar{\eta}(x_1) \dots \delta \bar{\eta}(x_n) \delta \eta(x'_1) \dots \delta \eta(x'_m)} U[\sigma_I, \sigma_{II}, \eta, J],$$

$$(3.6b) \quad \mathcal{G}_{\mu_1 \dots \mu_l}[\eta, J; z_1 \dots z_l] \equiv (i)^{-l} \frac{\delta^l}{\delta J_{\mu_1}(z_1) \dots \delta J_{\mu_l}(z_l)} U[\sigma_I, \sigma_{II}, \eta, J],$$

$$(3.6c) \quad g_{\mu_1 \dots \mu_l}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m, z_1 \dots z_l] \equiv \\ \equiv (i)^{-n-m-l} \frac{\delta^{n+m+l}}{\delta \bar{\eta}(x_1) \dots \delta \bar{\eta}(x_n) \delta \eta(x'_1) \dots \delta \eta(x'_m) \delta J_{\mu_1}(z_1) \dots \delta J_{\mu_l}(z_l)} U[\sigma_I, \sigma_{II}, \eta, J],$$

(*) Functional derivatives are defined by

$$F[\eta + \delta\eta, \bar{\eta} + \delta\bar{\eta}, J + \delta J] = F[\eta, \bar{\eta}, J] + \int \frac{\delta}{\delta \eta(\xi)} F[\eta, \bar{\eta}, J] \delta \eta(\xi) d^4\xi + \\ + \int \frac{\delta}{\delta \bar{\eta}(\xi)} F[\eta, \bar{\eta}, J] d^4\xi + \int \frac{\delta}{\delta J_\mu(\zeta)} F[\eta, \bar{\eta}, J] \delta J_\mu(\zeta) d^4\zeta,$$

with infinitesimal increments.

we obtain

$$(3.7a) \quad G[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m] \doteq \\ = \lambda^{n+m} \varepsilon(x_1 \dots x_n, x'_1 \dots x'_m) (\sigma_I, P[\psi(x_1) \dots \psi(x_n) \bar{\psi}(x'_1) \dots \bar{\psi}(x'_m)], \sigma_{II});$$

$$(3.7b) \quad \mathcal{G}_{\mu_1 \dots \mu_l}[\eta, J; z_1 \dots z_l] = \lambda'^l (\sigma_I, P[A_{\mu_1}(z_1) \dots A_{\mu_l}(z_l)], \sigma_{II});$$

$$(3.7c) \quad \mathbf{g}_{\mu_1 \dots \mu_l}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m, z_1 \dots z_l] = \\ = \lambda^{n+m} \lambda'^l \varepsilon(x_1 \dots x_n, x'_1 \dots x'_m) (\sigma_I, P[\psi(x_1) \dots \psi(x_n) \bar{\psi}(x'_1) \dots \bar{\psi}(x'_m) A_{\mu_1}(z_1) \dots A_{\mu_l}(z_l)], \sigma_{II}),$$

where P is the chronological operator (time increasing from right to left) and $\varepsilon(x_1 \dots x_n, x'_1 \dots x'_m)$ is the product of all $\varepsilon(x_i, x_j)$ ($i > j$), $\varepsilon(x'_i, x'_j)$ ($i > j$) and $\varepsilon(x_i, x'_j)$.

We shall denote the limit of a functional $F[\eta, J; x \dots]$ for $\bar{\eta} = \eta = 0$ or $J_\mu = 0$ by

$$F[J; x \dots] \equiv F[\eta = 0, J; x \dots]$$

$$F[\eta; x \dots] \equiv F[\eta, J = 0; x \dots]$$

$$F(x \dots) \equiv F[\eta = 0, J = 0; x \dots].$$

The functional Taylor expansion of U can now be written:

$$(3.7) \quad U[\sigma_I, \sigma_{II}, \eta, J] = 1 + \\ + \sum \frac{(i)^{n+m}}{n! m!} \int \bar{\eta}(\xi_1) \dots \bar{\eta}(\xi_n) G(\xi_1 \dots \xi_n, \xi'_1 \dots \xi'_m) \eta(\xi'_1) \dots \eta(\xi'_m) + \\ + \sum \frac{(i)^l}{l!} \int \mathcal{G}_{\mu_1 \dots \mu_l}(\zeta_1 \dots \zeta_l) J_{\mu_1}(\zeta_1) \dots J_{\mu_l}(\zeta_l) + \sum \frac{(i)^{n+m+l}}{n! m! l!} \\ \cdot \int \bar{\eta}(\xi_1) \dots \bar{\eta}(\xi_n) \mathbf{g}_{\mu_1 \dots \mu_l}(\xi_1 \dots \xi_n, \xi'_1 \dots \xi'_m, \zeta_1 \dots \zeta_l) \eta(\xi'_1) \dots \eta(\xi'_m) J_{\mu_1}(\zeta_1) \dots J_{\mu_l}(\zeta_l).$$

Here and in the following we integrate over all Greek variable (ξ, ζ) .

We give some properties of the vacuum expectation values of the derivatives defined in (3.7):

$$(3.8a) \quad G[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m] \equiv \langle 0 | G[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m] | 0 \rangle$$

$$(3.8b) \quad \mathcal{G}_{\mu_1 \dots \mu_l}[\eta, J; z_1 \dots z_l] \equiv \langle 0 | \mathcal{G}_{\mu_1 \dots \mu_l}[\eta, J; z_1 \dots z_l] | 0 \rangle$$

$$(3.8c) \quad \mathbf{g}_{\mu_1 \dots \mu_l}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m, z_1 \dots z_l] \equiv \\ \equiv \langle 0 | \mathbf{g}_{\mu_1 \dots \mu_l}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m, z_1 \dots z_l] | 0 \rangle.$$

Since the Lagrangian with $\eta = \bar{\eta} = 0$ is invariant under the transformation $\psi \rightarrow e^{i\alpha} \psi$, the functionals G and g must also be invariant under this transformation, therefore $G[\eta=0, J; \dots]$ and $g_{\mu_1 \dots}[\eta=0, J; \dots]$ with different numbers of ψ 's and $\bar{\psi}$'s, i.e. for $m \neq n$, are zero.

The Lagrangian is furthermore (for $\bar{\eta} = \eta = J_\mu = 0$) invariant under charge conjugation of the ψ -field together with $A_\mu \rightarrow -A_\mu$. Therefore the function $\mathcal{G}_{\mu_1 \dots}(\dots)$ made up of an odd number of A_μ 's must be zero. The following theorem permits us to call $G(x_1 \dots x_n, x'_1 \dots x'_n)$ and $\mathcal{G}_{\mu_1 \dots \mu_l}(z_1 \dots z_l)$ (with l even) propagators of electrons and photons, since they show the intimate relations between G , \mathcal{G} , and the probability-amplitudes.

Theorem I. The electron (photon) propagators

$$G(x_1 \dots x_n, x'_1 \dots x'_n)$$

and

$$\mathcal{G}_{\mu_1 \dots \mu_l \mu'_1 \dots \mu'_l}(z_1 \dots z_l, z'_1 \dots z'_l) \quad (l + l' = \text{even})$$

(where the undashed points are on σ_I , the dashed ones on σ_{II}) are proportional to the transition matrix elements between the state of m electrons (l photons) on σ_I , and n electrons (l' photons) on σ_{II} . Correspondingly, theorem II gives the physical interpretation of the mixed propagator.

Theorem II. The mixed propagators

$$g_{\mu_1 \dots \mu_l \mu'_1 \dots \mu'_l}(x_1 \dots x_n, x'_1 \dots x'_m, z_1 \dots z_l, z'_1 \dots z'_l)$$

where $x_1 \dots x_n, z_1 \dots z_l$ are on σ_I ; $x'_1 \dots x'_m, z'_1 \dots z'_l$ on σ_{II} are proportional to the transition matrix elements between n electrons on σ_I and n electrons on σ_{II} , accompanied by the emission of l photons on σ_I and the absorption of l' photons on σ_{II} .

We shall only prove the theorem I for $G(x, x')$. Generalization and the proof of II follow immediately. By picking out the Fourier amplitude corresponding to electron states with energies E' and E at the surfaces σ_{II} and σ_I respectively and taking into account the time order of σ_I, σ_{II} , we can write

$$G(x, x') \Big|_{\substack{x \text{ on } \sigma_I \\ x' \text{ on } \sigma_{II}}} = \lambda^2 \langle 0, \sigma_I | \bar{\psi}(x, E) \psi(x', E') | \sigma_{II}, 0 \rangle \exp [iEt - iE't'] .$$

We note further that the evolution operator, U , is the generating operator of the propagators as shown by the equations (3.6).

4. - Equations for the Propagators. Normalized Propagators.

Quite generally, the source part of the Hamiltonian has the form

$$\mathcal{H}_s(x) = Q(x)\alpha(x)$$

where $Q(x)$ is one of the field-quantities and $\alpha(x)$ represents its source. By generalising (3.5) we can write:

$$(4.1) \quad \frac{\delta}{\delta\alpha(x)} U[\sigma_I, \sigma_{II}, \alpha] = -i U[\sigma_I, \sigma, \alpha] Q(x) U[\sigma, \sigma_{II}, \alpha],$$

and find

$$(4.2) \quad \begin{aligned} \frac{\partial}{\partial x_\mu} \frac{\delta}{\delta\alpha(x)} U[\sigma_I, \sigma_{II}, \alpha] &= \frac{\delta}{\delta\sigma(x)} \int_{\sigma(x)} \frac{\delta}{\delta\alpha(x')} U[\sigma_I, \sigma_{II}, \alpha] d\sigma'_\mu = \\ &= -i \frac{\delta}{\delta\sigma(x)} \int_{\sigma(x)} U[\sigma_I, \sigma, \alpha] Q(x') U[\sigma, \sigma_{II}, \alpha] d\sigma'_\mu, \end{aligned}$$

$$(4.3) \quad \begin{aligned} &= -i U[\sigma_I, \sigma, \alpha] \frac{\partial}{\partial x_\mu} Q(x) U[\sigma, \sigma_{II}, \alpha] - \\ &- i \int_{\sigma(x)} \left(\frac{\delta}{\delta\sigma(x)} U[\sigma_I, \sigma, \alpha] \cdot Q(x') U[\sigma, \sigma_{II}, \alpha] + U[\sigma_I, \sigma, \alpha] Q(x') \frac{\delta}{\delta\sigma(x)} U[\sigma, \sigma_{II}, \alpha] \right) d\sigma'_\mu. \end{aligned}$$

Using (2.1) we obtain

$$(4.4) \quad \frac{\partial}{\partial x_\mu} \frac{\delta}{\delta\alpha(x)} U[\sigma_I, \sigma_{II}, \alpha] = -i \left(\sigma_I, \left[T_\mu + \int_{\sigma(x)} \mathcal{H}_s(x') d\sigma'_\mu, Q(x) \right], \sigma_{II} \right),$$

where T_μ is the energy-momentum vector in the source free representation.

When $Q(x)$ describes the electron field, i.e. $Q(x) = \psi(x)$ and $\alpha(x) = \lambda\bar{\eta}(x)$, we have, for $\mu = 4$,

$$\gamma_4 \frac{\partial}{\partial x_4} \frac{\delta}{\delta\bar{\eta}(x)} U[\sigma_I, \sigma_{II}, \eta, J] = i\lambda\gamma_4(\sigma_I[H + H_s, \psi(x)], \sigma_{II}),$$

with

$$[H + H_s, \psi(x)] = \gamma_4(-\mathbf{\Upsilon} \cdot \nabla - m + ie\gamma_\mu A_\mu(x))\psi(x) + \lambda\eta(x).$$

Since $\boldsymbol{\gamma} \cdot \nabla + m$ commutes with $U[\sigma, \sigma', \eta, J]$, we have

$$\gamma_4 \frac{\partial}{\partial x_4} \frac{\delta}{\delta \bar{\eta}(x)} U[\sigma_I, \sigma_{II}, \eta, J] = -i\lambda(\boldsymbol{\gamma} \cdot \nabla + m)(\sigma_I, \psi(x), \sigma_{II}) - \\ - e\lambda\gamma_\mu(\sigma_I, A_\mu(x)\psi(x), \sigma_{II}) + i\lambda^2\eta(x) U[\sigma_I, \sigma_{II}, \eta, J].$$

Taking into account formula (3.6), we finally obtain (*)

$$(4.5a) \quad \left(\gamma_\mu \partial_\mu + m - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)} \right) \frac{\delta}{\delta \bar{\eta}(x)} U[\sigma_I, \sigma_{II}, \eta, J] = i\lambda^2 \eta(x) U[\sigma_I, \sigma_{II}, \eta, J]$$

with

$$(4.6) \quad e_1 = e/\lambda'.$$

In a similar way we obtain, by taking $Q(x) = A_\mu(x)$ and $\alpha(x) = \lambda' J_\mu(x)$,

$$(4.5b) \quad \square \frac{\delta}{\delta J_\mu(z)} U[\sigma_I, \sigma_{II}, \eta, J] = i\lambda'^2 J_\mu(z) U[\sigma_I, \sigma_{II}, \eta, J] + \\ + i(\lambda'/\lambda)^2 e_1 \text{Sp} (G[J; z, z] \gamma_\mu).$$

Equations (4.5a) and (4.5b) are the generating equations for the many-body propagators; by operating successively with $\delta/\delta\eta$, $\delta/\delta\bar{\eta}$ and $\delta/\delta J_\mu$ we can derive equations for all possible propagators. This is done in the following for the one-electron, one-photon, and two-electron propagators.

Taking the vacuum expectation value of the derivative of (4.5a) with respect to $\eta(x')$, we obtain the equation for $G[\eta, J; x, x']$:

$$(4.7) \quad i \left(\gamma_\mu \partial_\mu + m - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)} \right) G[\eta, J; x, x'] = \\ = \lambda^2 \delta(x - x') \langle 0 | U[\sigma_I, \sigma_{II}, \eta, J] | 0 \rangle + i\lambda^2 \eta(x) \frac{\delta}{\delta \eta(x')} \langle 0 | U[\sigma_I, \sigma_{II}, \eta, J] | 0 \rangle.$$

Let us define the normalized propagators by dividing them by the vacuum expectation value of U , e.g.

$$(4.8a) \quad G_N[\eta, J; x, x'] \equiv \frac{G[\eta, J; x, x']}{\langle 0 | U[\sigma_I, \sigma_{II}, \eta, J] | 0 \rangle}.$$

It must be noted that the propagators are not changed by the above normal-

(*) A similar formula has been derived by SYMANZIK (7). We received his publication only after completion of this paper.

(7) K. SYMANZIK: *Zeits. Naturf.*, **9**, 809 (1954).

isation when η and J are zero, because of

$$\langle 0 | U[\sigma_I, \sigma_{II}, \eta=0, J=0] | 0 \rangle = 1.$$

Putting $\eta = 0$ in (4.7) we obtain the equation for the normalised propagator

$$(4.9) \quad i \left(\gamma_\mu \partial_\mu + m - ie_1 \gamma_\mu \mathcal{G}_{\mu,N}[J; x] - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)} \right) G_N[J; x, x'] = \lambda^2 \delta(x - x').$$

Application of $\delta^3/\delta\bar{\eta}(x_2) \delta\eta(x'_1) \delta\eta(x'_2)$ in (4.5a) gives the equation for the normalised two-electron propagator (with $\eta = 0$):

$$(4.10) \quad i \left(\gamma_\mu \frac{\partial}{\partial x_{1\mu}} + m - ie_1 \gamma_\mu \mathcal{G}_{\mu,N}[J; x_1] - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x_1)} \right) G_N[J; x_1 x_2 x'_1 x'_2] = \\ = \lambda'^2 \delta(x_1 - x'_1) G_N[J; x_2, x'_2] - \lambda'^2 \delta(x_1 - x'_2) G_N[J; x_2, x'_1].$$

In the same manner, we can deduce equations for the one-photon propagator from (4.5b):

$$(4.11) \quad i \square \mathcal{G}_{N,\mu\nu}(zz') = \lambda'^2 \delta(z - z') \delta_{\mu\nu} + (\lambda'/\lambda)^2 e_1 \text{Sp} \left\{ \gamma_\mu \frac{\delta}{\delta J_\nu(z')} G_N[J; z, z] \right\}_{J=0}.$$

5. - Vertex-Part, Self-Energy and Free Dressed Particle (*).

For the derivative of a composite functional we have

$$(5.1) \quad -i \frac{\delta}{\delta J_\mu(z)} G[J; x, x'] = -i \int \frac{\delta}{\delta \mathcal{G}_\nu[J; \zeta]} G[J; x, x'] \cdot \frac{\delta}{\delta J_\mu(z)} \mathcal{G}_\nu[J; \zeta] = \\ = \int \frac{\delta}{\delta \mathcal{G}_\nu[J; \zeta]} G[J; x, x'] \bar{\mathcal{G}}_{\nu\mu}[J; \zeta, z],$$

where

$$\bar{\mathcal{G}}_{\nu\mu}[J; z, z'] \equiv \mathcal{G}_{\nu\mu}[J; z, z'] - \mathcal{G}_\nu[J; z] \mathcal{G}_\mu[J; z'].$$

In order to express $\delta G/\delta \mathcal{G}_\nu$ by the vertex part Γ_ν , we assume the existence

(*) Since we always consider normalized propagators in the rest of the paper, we shall drop the index N .

of the inverse of the propagator $G[J; x, x']$ satisfying

$$(5.2) \quad \int G^{-1}[J; x, \xi] G[J; \xi, x'] = \delta(x - x')$$

and define the vertex-part by

$$(5.3) \quad \Gamma_v[J; z, x, x'] = \frac{1}{e_1} \frac{\delta}{\delta \mathcal{G}_v[J; z]} G^{-1}[J; x, x'],$$

$$(5.4) \quad = - \int G^{-1}[J; x, \xi] \cdot \frac{1}{e_1} \frac{\delta}{\delta \mathcal{G}_v[J; z]} G[J; \xi, \xi'] \cdot G^{-1}[J; \xi', x'].$$

Equation (5.4) leads to

$$(5.5) \quad \frac{1}{e_1} \frac{\delta}{\delta \mathcal{G}_v[J; z]} G[J; x, x'] = - \int G[J; x, \xi] \Gamma_v[J; z, \xi, \xi'] G[J; \xi', x'].$$

Finally, we have from (5.1)

$$(5.6) \quad i \frac{\delta}{\delta J_\mu(z)} G[J; x, x'] = e_1 \int G[J; x, \xi] \Gamma_v[J; z, \xi, \xi'] G[J; \xi', x'] \bar{\mathcal{G}}_{v\mu}[J; \xi, z].$$

Equations (5.5) and (5.6) have the graphical interpretations given in Fig. 1, where solid and wavy lines correspond to one-electron and one-photon propagators, respectively.

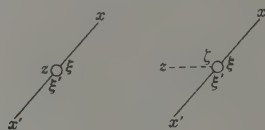


Fig. 1.



Fig. 2.

If $z = x$, this reduces to Fig. 2. Corresponding to Fig. 2, we introduce the mass-operator by

$$(5.7) \quad \mathcal{M}[J; x, x'] = ie_1^2 \int \gamma_\mu G[J; x, \xi] \Gamma_v[J; z, \xi, \xi'] \bar{\mathcal{G}}_{v\mu}[J; x, \xi].$$

Equation (5.6) goes over into

$$(5.8) \quad -e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)} G[J; x, x'] = \int \mathcal{M}[J; x, \xi] G[J; \xi, x'].$$

Then (4.9) can be written as follows:

$$(5.9) \quad i(\gamma_\mu \partial_\mu - ie_1 \gamma_\mu \mathcal{G}_\mu[J; x] + m) G[J; x, x'] + \\ + i \int \mathcal{M}[J; x, \xi] G[J; \xi, x'] = \lambda^2 \delta(x - x').$$

For $J_\mu =$ and $x \neq x'$, equation (4.5) goes over in

$$(5.10) \quad i(\gamma_\mu \partial_\mu + m) G(x, x') + i \int \mathcal{M}(x, \xi) G(\xi, x') = 0.$$

Since the theory is invariant under an inhomogeneous Lorentz transformation, the propagator $G(x, x')$ and the mass operator $\mathcal{M}(x, x')$ must be functions of $x - x'$. Their Fourier components depend on one momentum, p , only. In particular, the Fourier component of \mathcal{M} — this being a scalar — must be a function of $\gamma_\mu p_\mu$, we write it $\mathcal{M}(-i\gamma p)$.

Equation (5.10) reads then

$$[i\gamma_\mu p_\mu + m + \mathcal{M}(-i\gamma p)] G(p) = 0.$$

This is the equation of a particle with mass M , where $M = m^{(1)}, m^{(2)}, \dots$ is a root of the equation

$$(5.11) \quad \mathcal{M}(M) = M - m.$$

Taking in $G(x, x')$ the point x' on the initial surface σ_{II} , it follows that the incoming particles must have a spectrum $m^{(1)}, m^{(2)}, \dots$ of observable masses. We call these the free dressed particle. This is true also for outgoing particles (i.e. particles arriving on the surface σ_I).

6. — Two-Electron Propagator. I. Bethe-Salpeter Equation.

As in the case of the one-electron propagator, we introduce the inverse of the two-electron propagator and the two-electron vertex part by

$$(6.1a) \quad \int G[J; x_1 x_2, \xi_1 \xi_2] G^{-1}[J; \xi_1 \xi_2, x'_1 x'_2] = \\ = \frac{1}{2} \delta(x_1 - x'_1) \delta(x_2 - x'_2) - \frac{1}{2} \delta(x_1 - x'_2) \delta(x_2 - x'_1),$$

$$(6.1b) \quad \Gamma_\nu[J; z, x_1 x_2, x'_1 x'_2] = - \frac{\delta}{e_1 \delta \mathcal{G}_\nu[J; z]} G^{-1}[J; x_1 x_2, x'_1 x'_2].$$

From (6.1) we deduce as in section 5

$$(6.2) \quad e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(z)} G[J; x_1 x_2, x'_1 x'_2] = \\ = e_1^2 \gamma_\mu \int G[J; x_1 x_2, \xi_1 \xi_2] \Gamma_\nu[J; \zeta, \xi_1 \xi_2, \xi'_1 \xi'_2] G[J; \xi'_1 \xi'_2, x'_1 x'_2] \bar{\mathcal{G}}_{\nu\mu}[J; \zeta z].$$

Further, we introduce a two-electron mass operator in the following way:

$$(6.3) \quad \int G[J; x_2, \xi] \mathcal{M} J; x_1 \xi, x'_1 x'_2] = -\frac{1}{2} \{ \mathcal{M}[J; x_1, x'_1] \delta(x_2 - x'_2) - \\ - \mathcal{M}[J; x_1, x'_2] \delta(x_2 - x'_1) \} + e_1^2 \gamma_\mu \int G[J; x_1 x_2, \xi_1 \xi_2] \Gamma_\nu[J; \zeta, \xi_1 \xi_2, x'_1 x'_2] \bar{\mathcal{G}}_{\nu\mu}[J; x_1 \zeta],$$

which leads to

$$(6.4) \quad -e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x_1)} G[J; x_1 x_2, x'_1 x'_2] = \int \mathcal{M}[J; x_1, \xi] G[J; \xi x_2, x'_1 x'_2] + \\ + \int G[J; x_2, \xi_2] \mathcal{M}[J; x_1 \xi_2, \xi'_1 \xi'_2] G[J; \xi'_1 \xi'_2, x'_1 x'_2].$$

By substituting (6.4) into (4.10) we can derive the Bethe-Salpeter equation:

$$(6.5) \quad G(x_1 x_2, x'_1 x'_2) = G(x_1, x'_1) G(x_2, x'_2) - G(x_1, x'_2) G(x_2, x'_1) - \\ - \int G(x_1, \xi_1) G(x_2, \xi_2) \mathcal{M}(\xi_1 \xi_2, \xi'_1 \xi'_2) G(\xi'_1 \xi'_2, x'_1 x'_2).$$

7. - Two-Electron Propagator. II. The Renormalized Equations.

In this section, we shall show that the equation (4.10) of the two-electron propagator is equivalent to the following set:

$$(7.1a) \quad G(x_1 x_2, x'_1 x'_2) = G(x_1, x'_1) G(x_2, x'_2) - \\ - i e_1 \int G(x_1, \xi) v_e(\zeta, \xi, \xi') I_e(\zeta, \xi' x_2, x'_1 x'_2) - (x'_1 \rightarrow x'_2),$$

where

$$(7.1b) \quad v_\mu[J; z, x, x'] \equiv i \Gamma_\mu[J; \zeta, x, \xi] G[J; \xi, x'] \bar{\mathcal{G}}_{e\mu}[J; \zeta z]$$

and \hat{I}_μ has to satisfy the equation

$$(7.1c) \quad I_\mu[J; z, x_1 x_2, x'_1 x'_2] = \delta(x_1 - x'_1) \frac{\delta}{\delta \mathcal{G}_\mu[J; z]} G[J; x_2, x'_2] + \\ + e_1 \int V_{\mu e}[J; z, x_1, \zeta, \xi_1] I_e[J; \zeta, \xi_1 x_2, x'_1 x'_2],$$

with

$$(7.1d) \quad V_{\mu e}[J; z, x, z', x'] = v_e[J; z', x, x'] \frac{\delta}{\delta \mathcal{G}_\mu[J; z]}.$$

Here the bracket $(x'_1 \rightarrow x'_2)$ means the addition of all the terms obtained by exchanging x'_1 and x'_2 . We call (7.1) the «renormalised equations» because it can be shown (cf. sections 8 and 9) that they can yield the renormalised solutions.

Before deducing (7.1), let us consider its physical meaning. Successive approximation of (7.1b) leads to

$$(7.2) \quad I_\mu[J; z, x_1 x_2, x'_1 x'_2] = \\ = -e_1 \delta(x_1 - x'_1) \int G[J; x_2, \xi_2] I_\mu[J; z, \xi_2, \xi'_2] G[J; \xi'_2, x'_2] + \\ + \sum_1 (-e_1)^{n+1} \int V_{\mu e_n}[J; z, x_1, \zeta_n, \xi_n] V_{e_n e_{n-1}}[J; \zeta_n, \xi_n, \zeta_{n-1}, \xi_{n-1}] \dots \\ \dots V_{e_2 e_1}[J; \zeta_2, \xi_2, \zeta_1, x'_1] G[J; x_2, \xi'] I_{e_1}[J; \zeta_1, \xi', \xi] G[J; \xi, x'_2].$$

By substituting this expansion into (7.1a) we obtain the two-electron propagator which can be graphically expressed by Fig. 3.

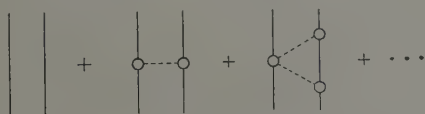


Fig. 3.

The n -th term of this expansion can be represented by Fig. 4, in which a photon created at a vertex Γ_n can be absorbed by electron lines, photon lines and vertices only in the region above the boundary represented by the dotted curve $n-2$. It can be seen from Fig. 4 that we have no self-energy and corrected vertex diagrams. Therefore, diagram 4 is made up of all possible irreducible diagrams (i.e. skeletons).

Alternatively, the Fredholm method can be applied ⁽⁸⁾ for solving (7.1c). Let us denote by K the integral operator with the kernel (7.1d) and define the integral operators \mathcal{D}_n by the following formula:

$$(7.3) \quad \mathcal{D}_{n+1} = K\mathcal{D}_n - nK\mathcal{D}_n$$

with

$$(7.4) \quad D_n \equiv \text{Sp } \mathcal{D}_n = \int \langle \xi | \mathcal{D}_n | \xi \rangle d^4\xi, \quad D_0 \equiv 1.$$

The solution I_μ of equation (7.1c) is

$$(7.5) \quad I_\mu(z, x_1x_2, x'_1x'_2) = \left[1 + \frac{\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \mathcal{D}_{n+1}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} D_n} \right] \delta(x_1 - x'_1) \delta_{\mathcal{G}_\mu[J; z]} G[J; x_2, x'_2]_{J=0}.$$

It can be shown that the term \mathcal{D}_{n+1} in the numerator of (7.5) includes all the diagrams with a number of vertices up to n as shown by equation (7.3). The denominator of (7.5) is made up of closed diagrams (i.e. vacuum diagrams) given in Fig. 5.

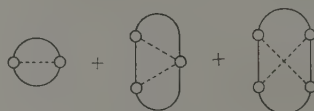


Fig. 5.

Thus, by substituting renormalised one-body propagators and renormalised vertices in the equations (7.1) we obtain the renormalised solution automatically. This is the essential advantage of the equations (7.1).

The derivation of equation (7.1) is as follows: Equation (4.9) gives

$$\left(\gamma_e \hat{c}_e + m + ie_1 \gamma_e \mathcal{G}_e[J; x] - e_1 \gamma_e \frac{\delta}{\delta J_e(x)} \right) \frac{\delta}{\delta J_\mu(z)} G[J; x, x'] = e_1 \gamma_e \bar{\mathcal{G}}_{e\mu}[J; xz] G[J; x, x'],$$

which leads, using equations (5.6), to

$$(7.6) \quad e_1 \gamma_e \bar{\mathcal{G}}_{e\mu}[J; xz] G[J; x, x'] = -\lambda^2 e_1 \int \Gamma_e[J; \zeta, x, \xi] \bar{\mathcal{G}}_{e\mu}[J; \zeta z] G[J; \xi, x'] + ie_1 \gamma_e \int G[J; x, \xi] \frac{\delta}{\delta J_e(x)} \{ \Gamma_e[J; \zeta, \xi, \xi'] G[J; \xi', x'] \bar{\mathcal{G}}_{e\mu}[J; \zeta z] \}.$$

We can show by means of (7.1c) and (7.6) that (7.1a) satisfies (4.10).

⁽⁸⁾ A. VISCONTI: *Journ. Phys. et Rad.*, **16**, 1 (1955).

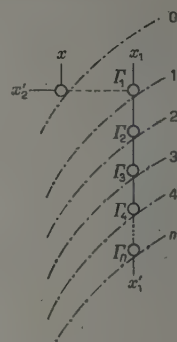


Fig. 4.

This method can easily be extended to other many body propagators. For instance, the propagator for the one-electron-one-photon system (Compton scattering) can be derived from (4.5a) as follows:

$$(7.7) \quad i \left(\gamma_e \partial_e + m - ie_1 \gamma_e \mathcal{G}_e[J; x] - e_1 \gamma_e \frac{\delta}{\delta J_{\square}(x)} \right) g_{\mu\nu}[J; x, x', z_1 z_2] = \\ = \lambda^2 \delta(x - x') (\bar{\mathcal{G}}_{\mu\nu}[J; z_1 z_2] - \mathcal{G}_{\mu}[J; z_1] \mathcal{G}_{\nu}[J; z_2]) .$$

We can rewrite (7.7) by means of (7.6) in the following way:

$$(7.8a) \quad g_{\mu\nu}(x, x', z_1 z_2) = G(x, x') \bar{\mathcal{G}}_{\mu\nu}(z_1 z_2) + e_1 \int G(x, \xi') v_{\sigma}(\zeta, \xi', \xi) I_{\sigma\mu\nu}(\zeta, \xi, x', z_1 z_2),$$

where $I_{\sigma\mu\nu}[\dots]$ is a solution of

$$(7.8c) \quad I_{\sigma\mu\nu}[J; z, x, x', z_1 z_2] = \delta(x - x') \left\{ \frac{\delta}{\delta \mathcal{G}_{\sigma}[J; z]} \bar{\mathcal{G}}_{\mu\nu}[J; z_1 z_2] + \right. \\ \left. + \delta(z - z_1) \delta_{\sigma\mu} \mathcal{G}_{\nu}[J; z_2] + \delta(z - z_2) \delta_{\sigma\nu} \mathcal{G}_{\mu}[J; z_1] \right\} + \\ + e_1 \int V_{\sigma\varrho}[J; z, x, \zeta, \xi] I_{\varrho\mu\nu}[J; \zeta, \xi, x', z_1 z_2].$$

These equations give directly the renormalised matrix elements of the Compton scattering. This is an essential difference between equation (7.8) and the Bethe-Salpeter equation, in which the ladder-approximation gives rise to new infinities.

8. - Renormalization Invariance.

The transformation

$$(8.1) \quad \lambda \rightarrow c\lambda, \quad \lambda' \rightarrow c'\lambda', \quad e_1 \rightarrow e_1 c'$$

induces the following changes.

$$(8.2) \quad G(x, x') \rightarrow c^2 G(x, x'), \quad \bar{\mathcal{G}}_{\mu\nu}(zz') \rightarrow c'^2 \bar{\mathcal{G}}_{\mu\nu}(zz')$$

and analogous changes for the many-body propagators. We call (8.1) the *renormalisation transformation* because it changes the normalisation factors of the propagators.

We note that equations (7.1) and (7.8) for Moller and Compton scattering

contain neither λ and λ' and are therefore invariant under the transformation (8.1); they are *renormalisation invariant*. It is physically reasonable that the value of the charge depends on the normalisation of the propagators; indeed if the strength of the electromagnetic field is multiplied by e' the charge must be divided by the same constant. Then, there remains one condition to be fulfilled by the renormalisation constants: the eigenvalues of the number operator for the photon must be integers.

9. - The Normalization Constants.

Let us consider the equation (5.9) of the one-electron propagators, for $J=0$. Its Fourier transform has the form

$$(9.1) \quad i(\gamma_\mu p_\mu + m + \mathcal{M}(-i\gamma p))G(p) = \lambda^2.$$

The causality principle requires

$$G(x, x') = \begin{cases} G^-(x - x') & \text{for } t < t' \\ G^+(x - x') & \text{for } t > t'. \end{cases}$$

Such a propagator is obtained from (9.1) as follows:

$$(9.2) \quad G(x, x') = \frac{\lambda^2}{(2\pi)^4} \int_L d^4p \exp[i(p, x - x')] \frac{1}{i\gamma_\mu p_\mu + m + \mathcal{M}(-i\gamma p)}.$$

The integration path L is that used in the definition of the S_F function. From (5.11) we see that the denominator can be written in the form

$$(9.3) \quad a(-i\gamma p) \prod_{j=1} (i\gamma_\mu p_\mu + m^{(j)}),$$

where $m^{(1)} \leq m^{(2)} \leq \dots$ and a is a function without zero. To each eigenvalue $m^{(j)}$ of the mass we define a constant $Z_2^{(j)}$ by

$$(9.4) \quad \frac{1}{Z_2^{(j)}} \equiv \frac{\partial}{\partial M} (M - \mathcal{M}(M)) \Big|_{M=m^{(j)}} = a(m^{(j)}) \prod_{k \neq j} (m^{(k)} - m^{(j)}).$$

Then (9.2) can be written in the form

$$(9.6) \quad G(x, x') = - \frac{i\lambda^2}{(2\pi)^4} \sum_{j=1} Z_2^{(j)} \int_L d^4p \exp[i(p, x - x')] \frac{1}{i\gamma_\mu p_\mu + m^{(j)}} \cdot \frac{a(m^{(j)})}{a(-i\gamma p)} =$$

$$= -\frac{i\lambda^2}{(2\pi)^4} \sum_{j=1} Z_2^{(j)} \int_L d^4p \exp[i(p, x-x')] \left[P \frac{1}{i\gamma_\mu p_\mu + m^{(j)}} \cdot \frac{a(m^{(j)})}{a(-i\gamma p)} - \right. \\ \left. - i\pi(i\gamma_\mu p_\mu - m^{(j)}) \delta(p_\nu p_\nu + m^{(j)2}) \right].$$

The second term in the bracket of (9.6) shows that the number operator of the particle with mass $m^{(j)}$ appears with the normalization factor $\lambda^2 Z_2^{(j)}$ and has therefore the eigenvalues

$$\lambda^2 Z_2^{(j)}, \quad 2\lambda^2 Z_2^{(j)}, \quad \dots$$

Equation (9.5) shows that some of the $Z_2^{(j)}$ may be negative. That is certainly the case if $a(-i\gamma p)$ is an holomorphic function; its sign must then be definite, since it has no zero. The sign of the product in (9.5) and so of $Z_2^{(j)}$ alternates with increasing j .

In the case of a field with more than one mass level (*) we are faced with a difficulty. It is impossible to make all normalisation factors $\lambda^2 Z_2^{(j)}$ equal to 1 by adjusting the one constant, λ .

In the following discussion, we shall assume that the electron has only one mass level $m^{(1)}$. Then, for the renormalization of the number operator of the dressed electron we have to take (with $Z_2 = Z_2^{(1)}$)

$$(9.7) \quad \lambda^2 = 1/Z_2.$$

From (9.3) and (9.5) we obtain

$$(9.8) \quad i\gamma_\mu p_\mu + m + \mathcal{M}(-i\gamma p) = (1/Z_2)(i\gamma_\mu p_\mu + m^{(1)})(1 + \mathcal{M}_1(-i\gamma p)),$$

where, according (9.3),

$$(9.9) \quad \mathcal{M}_1(-i\gamma p) = \frac{a(-i\gamma p)}{a(m^{(1)})} - 1,$$

so that

$$\mathcal{M}_1(m^{(1)}) = 0.$$

Therefore

$$(9.10) \quad G(x, x') = \frac{1}{(2\pi)^4} \int_L d^4p \exp[i(p, x-x')] \frac{1}{i\gamma_\mu p_\mu + m^{(1)}} \cdot \frac{1}{1 + \mathcal{M}_1(-i\gamma p)}.$$

(*) LEE (9) gave a simple model of interacting fields, which leads to two mass levels.

(9) T. D. LEE: *Phys. Rev.*, **95**, 1329 (1954).

The one-photon propagator can be discussed in a similar way. Gauge invariance admits of only one mass, namely zero. The normalization constant λ' can be determined in a similar way to λ .

10. — Renormalized Theory.

The aim of this section is to prove that the present theory of propagators is a renormalized theory. For this purpose we shall calculate

$$\frac{1}{e_1} \frac{\delta}{\delta \mathcal{G}_\mu[J; z]} F[J; z_1 \dots z_l, x_1 \dots x_n, x'_1 \dots x'_n]$$

for the limit of vanishing momentum of the photon at z . (As usual z and x denote photon and electron coordinate respectively.)

Let

$$F[J; z_1 \dots z_l, p_1 \dots p_n, p'_1 \dots p'_n]$$

be the Fourier component of the functional

$$F[J; z_1 \dots z_l, x_1 \dots x_n, x'_1 \dots x'_n],$$

i.e.

$$\begin{aligned} (10.1) \quad F[J; z_1 \dots z_l, x_1 \dots x_n, x'_1 \dots x'_n] = \\ = (2\pi)^{-sn} \int F[J; z_1 \dots z_l, p_1 \dots p_n, p'_1 \dots p'_n] \exp \left[i \left\{ \sum_{j=1}^n (p_j x_j - p'_j x'_j) \right\} \right], \end{aligned}$$

and be

$$F_\mu[J; z_1 \dots z_l, p_1 \dots p_n, p'_1 \dots p'_n]$$

the Fourier component of

$$\frac{1}{e_1} \frac{\delta}{\delta \mathcal{G}_\mu[J; z]} F[J; z_1 \dots z_l, x_1 \dots x_n, x'_1 \dots x'_n]$$

for the case of vanishing momentum of the photon at z . Then:

$$\begin{aligned} (10.2) \quad \frac{1}{e_1} F_\mu[J; z_1 \dots z_l, p_1 \dots p_n, p'_1 \dots p'_n] = \\ = (2\pi)^{-4} \sum_{j=1}^n \left\{ \frac{\partial}{\partial p_{j\mu}} - \frac{\partial}{\partial p'_{j\mu}} \right\} F[J; z_1 \dots z_l, p_1 \dots p_n, p'_1 \dots p'_n]. \end{aligned}$$

The formula (10.2) can be proved by taking into account the fact that $\mathcal{G}_\mu[J; Z]$ appears in the form $\partial_\mu - ie_1 \mathcal{G}_\mu$ in the equations for the propagators, and that the derivatives $\partial_\nu \mathcal{G}_\mu$, $\partial_\nu \partial_\nu \mathcal{G}_\mu$, ... can be neglected in the limiting case of vanishing momentum of the photon at z .

From (9.1), (9.3) and (9.10), we can derive

$$(10.3) \quad (i\gamma_\mu p_\mu + m^{(1)})(1 + \mathcal{M}_1(-i\gamma p)) G(p) = 1,$$

where $m^{(1)}$ is the experimental mass. Now (9.8) shows that $\mathcal{M}_1(-i\gamma p)$ and Z_2 correspond respectively to the finite part of the mass-operator Σ^* , and to Z_2 , in the Dyson theory. Therefore, we see from (10.3) that $G(p)$ is already renormalized.

Formulae (5.3) and (10.2) give

$$(10.4) \quad \Gamma_\mu(l=0, p) = \gamma_\mu + \frac{\partial}{\partial p_\mu} \{ (\gamma^\mu p_\mu + m^{(1)}) \mathcal{M}_1(-i\gamma p) \},$$

where l and p are the energy-momentum vectors of photon and electron. This relation shows that Γ_μ is the renormalised vertex part, because Dyson proved that $\Gamma_\mu(l, p)$ is renormalised for any value of l when it is so for $l=0$. Since we showed in section 7 that equations (7.1) give only *irreducible diagrams* made up of corrected lines and vertices and the observable charge e_1 , we may conclude that the theory is renormalized.

11. - A Method of Solution.

Since the integral equations (7.1) and (7.8) which describe respectively Moller and Compton scattering, have the same kernel $V_{\mu\nu}$, they can be solved by using the common resolvent operator $R_{\mu\nu}[\mathcal{G}, \delta/\delta\mathcal{G}, z, x, z', x']$. Both I_μ and $I_{\lambda\mu\nu}$ can be expressed by $R_{\mu\nu}$:

$$(11.1) \quad I_\mu(z, x_1 x_2, x'_1 x'_2) = -e_1 \delta(x_1 - x'_1) \int G(x_2, \xi) \Gamma_\mu(z, \xi, \xi') G(\xi', x'_2) - \\ - e_1 \int R_{\mu 0} \left[\mathcal{G}, \frac{\partial}{\partial \mathcal{G}}, z, x_1, \xi, x'_1 \right] G[J; x_2, \xi] \Gamma_\nu[J; \xi, \xi, \xi'] G[J; \xi', x'_2] \Big|_{J=0}.$$

$$(11.2) \quad I_{\lambda\mu\nu}(z, x, x', z_1 z_2) = \int R_{\lambda 0} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}_0}, z, x, \xi, x' \right] \frac{\delta}{\delta \mathcal{G}_0[J; \xi]} \mathcal{G}_{\mu\nu}[J; z_1 z_2] \Big|_{J=0} + \\ + \int R_{\lambda\mu} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}}, z, x, z_1, x' \right] \mathcal{G}_\nu[J; z_2] \Big|_{J=0} + \\ + \int R_{\lambda\nu} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}}, z, x, z_2, x' \right] \mathcal{G}_\mu[J; z_1] \Big|_{J=0}.$$

$R_{\mu\nu}$ satisfies the following integro-differential equation corresponding to the resolvent equation:

$$(11.3) \quad R_{\mu\nu} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}}, z, x, z', x' \right] = V_{\mu\nu} [J; z, x, z', x'] + \\ + \int V_{\mu\nu} [J; z, x, \zeta, \xi] R_{\nu\sigma} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}} \zeta, \xi, z', x' \right].$$

We now give a solution of (11.3) by limiting the number of bosons in flight. Since according to (11.3) $R_{\mu\nu}$ is a functional of \mathcal{G} and $\delta/\delta \mathcal{G}$, we can make a functional Taylor expansion with respect to $\delta/\delta \mathcal{G}$:

$$(11.4) \quad R_{\mu\nu} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}} z, x, z', x' \right] = \\ = \sum_n \int R_{\mu\nu\varrho_1 \dots \varrho_n}^{(n)} [\mathcal{G}, z, x, z', x'; \zeta_1 \dots \zeta_n] \frac{\delta}{\delta \mathcal{G}_{\varrho_1} [J; \zeta_1]} \dots \frac{\delta}{\delta \mathcal{G}_{\varrho_n} [J; \zeta_n]}.$$

By substituting (11.4) into (11.3) and by picking up terms of the same order in $\delta/\delta \mathcal{G}$, we obtain a recurrent set of equations

$$(11.5) \quad R_{\mu\nu\varrho_1 \dots \varrho_n}^{(n)} [\mathcal{G}, z, x, z', x'; z_1 \dots z_n] = \\ = i \int v_\sigma [J; \zeta, x, \xi] \frac{\delta}{\delta \mathcal{G}_\mu [J; z]} R_{\sigma\nu\varrho_1 \dots \varrho_n}^{(n)} [\mathcal{G}, \zeta, \xi, z', x'; z_1 \dots z_n] + \\ + i \int v_\sigma [J; \zeta, x, \xi] R_{\sigma\nu\varrho_1 \dots \varrho_{n-1}}^{(n-1)} [\mathcal{G}, \zeta, \xi, z', x'; z_1 \dots z_{n-1}] \delta_{\mu\varrho_n} \delta(z - z_n).$$

with v_σ from (7.1b). In the case $n=1$ an extra term has to be added (see (11.7)). For $n=0, 1$ we obtain so

$$(11.6) \quad R_{\mu\nu}^{(0)} [\mathcal{G}, z, x, z', x''] = i \int v_\sigma [J; \zeta, x, \xi] \frac{\delta}{\delta \mathcal{G}_\mu [J; z]} R_{\sigma\nu}^{(0)} [\mathcal{G}, \zeta, \xi, z', x'],$$

$$(11.7) \quad R_{\lambda\mu\nu}^{(1)} [\mathcal{G}, z, x, z', x'] = i \int v_\sigma [J; \zeta, x, \xi] \frac{\delta}{\delta \mathcal{G}_\lambda [J; z]} R_{\sigma\mu\nu}^{(1)} [\mathcal{G}, \zeta, \xi, z', x'; z_1] + \\ + i \delta_{\lambda\nu} \delta(z - z_1) \int v_\sigma [J; \zeta, x, \xi] R_{\sigma\mu}^{(0)} [\mathcal{G}, \zeta, \xi, z', x'] + \\ + i \delta_{\lambda\nu} \delta(z - z_1) v_\mu [J; z', x, x'].$$

Since $R^{(0)}$ satisfies the homogeneous integral equation (11.6), $R^{(0)}=0$ is a

possible solution. For scattering problems, this is the correct solution. One obtains namely, in this case, by successive approximations all possible irreducible diagrams by starting from $-R^{(0)}=0$.

To calculate $R^{(1)}$ from (11.7) for the case of scattering, we expand $R_{\lambda\mu\nu}^{(1)}$ and v_μ in powers of \mathcal{G}_μ :

$$(11.8) \quad v_\mu[J; z, x, x'] = \sum_m v_{\mu q_1 \dots q_m}^{(m)}(z, x, x'; \zeta_1 \dots \zeta_m) \mathcal{G}_{q_1}[J; \zeta_1] \dots \mathcal{G}_{q_m}[J; \zeta_m]$$

$$(11.9) \quad R_{\lambda\mu\nu}^{(1)}[\mathcal{G}; z, x, z', x'; z_1] = \\ = \sum_m \int R_{\lambda\mu\nu q_1 \dots q_m}^{(1, m)}(z, x, z', x'; z_1; \zeta_1 \dots \zeta_m) \mathcal{G}_{q_1}[J; \zeta_1] \dots \mathcal{G}_{q_m}[J; \zeta_m].$$

Since the appearance of a functional derivative $\delta/\delta\mathcal{G}$ corresponds to the insertion of a photon line, we may limit the number of virtual photons by cutting off the development. As a simple example of this approximation method, we shall consider the case where we take into account only one such photon. That corresponds to neglecting all $R^{(m, n)}$ and $v^{(m)}$ with exception of $R^{(1, 0)}$, $R^{(1, 1)}$, $v^{(0)}$, $v^{(1)}$. Then equation (11.7) leads to

$$(11.10) \quad R_{\lambda\mu\nu}^{(1, 0)}(z, x, z', x'; z_1) = i\delta_{\lambda\nu} \delta(z - z_1) v_\mu^{(0)}(z', x, x') + \\ + i \int v_\rho^{(0)}(\zeta, x, \xi) R_{\rho\mu\nu\lambda}^{(1, 1)}(\zeta, \xi, z', x'; z_1; z);$$

$$(11.11) \quad R_{\lambda\mu\nu\rho}^{(1, 1)}(z, x, z', x'; z_1; z_2) = i\delta_{\lambda\nu} \delta(z - z_1) v_{\mu\rho}^{(1)}(z', x, x'; z_2) - \\ - \int v_{\sigma\rho}^{(1)}(\zeta, x, \xi; z_2) R_{\sigma\mu\nu\lambda}^{(1, 1)}(\zeta, \xi, z', x'; z_1; z).$$

Furthermore, we assume for simplicity

$$(11.12) \quad \begin{cases} G(x, x') = S_F(x - x') \\ \bar{\mathcal{G}}_{\mu\nu}(zz') = \delta_{\mu\nu} D_F(z - z') \\ \Gamma_\mu(z, x, x') = \gamma_\mu \delta(z - x) \delta(z - x'). \end{cases}$$

Then

$$(11.13) \quad v_\mu^{(0)}(z, x, x') = \gamma_\mu D_F(x - z) S_F(x - x')$$

$$(11.14) \quad v_{\mu\nu}^{(1)}(z, x, x'; z_1) = -e_1 \gamma_\mu D_F(x - z) S_F(x - z_1) \gamma_\nu S_F(z_1 - x').$$

The writers would like to thank Professor ROSENFELD for inspiring discussions and his kind hospitality. Thanks are also due to Dr. J. PODOLANSKI for many stimulating comments.

RIASSUNTO (*)

Si sviluppa una teoria generale dei propagatori. Tutti i propagatori si derivano da un funzionale generatore. La rinormalizzazione corrisponde all'adattamento delle costanti di accoppiamento delle sorgenti esterne. Come esempio si derivano delle equazioni per gli scattering di Compton e di Møller.

(*) Traduzione a cura della Redazione.

Generalized Beta Invariants.

R. J. FINKELSTEIN

Department of Physics, University of California - Los Angeles

(ricevuto il 21 Marzo 1955)

Summary. — A wave equation covariant under conformal transformations is proposed. Generalized beta invariants are constructed from the eight rowed wave functions which satisfy this equation, and some of their formal properties are derived.

1. — Introduction.

In this paper we give a formal generalization of the beta interactions based on the conformal ⁽¹⁾ instead of the Lorentz group. The possible very speculative connection between these formal results and the observed beta couplings will be discussed in a separate paper.

Since conformal transformations do not preserve length, they do not preserve rest mass in momentum space, and the four components of the energy momentum vector (p_k) must therefore be regarded as independent. The subsequent analysis will be based on the six variables P_μ which are defined in terms of the four components as follows:

$$(1) \quad \left\{ \begin{array}{l} \varrho P_k = p_k \\ \varrho P_5 = (m^2 + \sum_1^4 p_k^2)/2im \\ \varrho P_6 = (m^2 - \sum_1^4 p_k^2)/2m, \end{array} \right. \quad k = 1, 2, 3, 4$$

⁽¹⁾ For references to the literature on the possible role of the conformal group in physics see R. L. INGRAHAM: *Nuovo Cimento*, **12**, 826 (1954) and Y. MURAI: *Prog. Theor. Phys.*, **11**, 441 (1954).

where m is a constant with the dimensions of a mass. After ϱ is fixed, the four p_k are expressed in terms of the six P_μ ; hence there are two relations between the P_μ :

$$(2a) \quad \sum_1^6 P_\mu^2 = 0$$

$$(2b) \quad P_5 - iP_6 = -im/\varrho.$$

The conformal transformations of p_k are rotations and reflexions of the P_μ .

In this paper we are interested in a wave equation which is form invariant under the rotations and reflexions of the P_μ and in the interactions between particles described by such equations. The equation may be regarded as a generalization of the Dirac equation and the interactions to which it leads may be considered as generalizations of the usual beta interactions.

Denote the classical motion by P_μ and the corresponding quantum state by $\psi(P)$. It is now postulated that the fundamental wave equation may be obtained by Dirac factorization of (2a):

$$(3a) \quad \sum_1^6 \Gamma_\mu P_\mu \psi = 0$$

$$(3b) \quad (\Gamma_\mu, \Gamma_\nu)_+ = 2\delta_{\mu\nu}.$$

One is interested in the transformation properties of ψ when the P are subject to rotations and reflections. This is the same problem as one solves for the usual Dirac equation. In the following paragraphs the usual analysis will be generalized to the even n -dimensional case.

2. - Commutation Properties.

Let there be $n = 2k$ anticommuting γ_μ : $[\gamma_\mu, \gamma_\nu]_+ = 2\delta_{\mu\nu}$ which are hermitian: $\gamma_\mu = \gamma_\mu^+$. Choose the γ_μ to be either pure real or pure imaginary and let the number of imaginary ones be p . We denote an arbitrary product of σ factors by I^σ .

$$I^\sigma = \gamma_i \gamma_j \dots \gamma_k \equiv \gamma_{ij} \dots k.$$

Then I^σ is unitary:

$$(1) \quad (I^\sigma)^+ = \gamma_{k \dots j i} = (I^\sigma)^{-1}$$

and has the property

$$(2) \quad (I^\sigma)^+ = \gamma_{k \dots j i} = (-)^{\sigma-1} (-)^{\sigma-2} \dots (-)^1 \gamma_{ij \dots k} = \varepsilon(\sigma) I^\sigma$$

where

$$\varepsilon(\sigma) = (-)^{\sigma(\sigma-1)/2}$$

In particular let C be the product of the imaginary matrices only: $\gamma_1 \dots \gamma_p$. Then

$$(3) \quad C^+ = C^{-1} = \varepsilon(p)C$$

$$(4) \quad \tilde{C} = (-)^p \varepsilon(p)C$$

where \sim means transpose. One has

$$C^{-1} \gamma_\mu C = \gamma_{p \dots 1} \gamma_\mu \gamma_{1 \dots p}.$$

Therefore if γ_μ is real

$$C^{-1} \gamma_\mu C = (-)^p \gamma_\mu$$

and if γ_μ is imaginary

$$C^{-1} \gamma_\mu C = (-)^{p+1} \gamma_\mu.$$

In either case

$$(5) \quad C^{-1} \gamma_\mu C = (-)^p \gamma_\mu^* = (-)^p \tilde{\gamma}_\mu.$$

Similarly

$$C^{-1} \Gamma^\sigma C = C^{-1} \gamma_{i \dots k} C = (-)^{p\sigma} \tilde{\gamma}_i \dots \tilde{\gamma}_k = (-)^{p\sigma} \tilde{\gamma}_{k \dots i}.$$

Therefore

$$(6a) \quad C^{-1} \Gamma^\sigma C = (-)^{p\sigma} \varepsilon(\sigma) \tilde{\Gamma}^\sigma$$

$$(6b) \quad C^{-1} \Gamma^\sigma C = (-)^{p\sigma} \varepsilon(\sigma) (-)^t \Gamma^\sigma \equiv \lambda \Gamma^\sigma$$

where

$$(6c) \quad \lambda = (-)^{p\sigma} \varepsilon(\sigma) (-)^t,$$

where t is the number of the imaginary factors in Γ^σ .

In the following we shall be particularly interested in products of the form

$$E^\sigma = C^{-1} \Gamma^\sigma.$$

These matrices are either symmetric or antisymmetric. For

$$\tilde{E}^\sigma = \tilde{\Gamma}^\sigma \tilde{C}^{-1} = [(-)^{p\sigma} \varepsilon(\sigma) (C^{-1} \Gamma^\sigma C)] [(-)^p \varepsilon(p) C^{-1}]$$

by (4) and (6a). Hence

$$(7) \quad \tilde{E}^\sigma = (-)^{p\sigma} \varepsilon(\sigma) (-)^p \varepsilon(p) E^\sigma$$

so that the symmetry of E depends on (p, σ) .

It is also convenient to introduce the symbols E_ϱ^σ and Γ_ϱ^σ as follows:

$$(8) \quad E_\varrho^\sigma = C^{-1} \Gamma_\varrho^\sigma,$$

where

$$(9) \quad \Gamma_\varrho^\sigma = \varepsilon_\varrho^\sigma \gamma_1^{s_1} \dots \gamma_n^{s_n}.$$

Here the exponents are restricted to be either 0 or 1 and in terms of them

$\sigma = \sum_{k=1}^n s_k$. Those γ_k which appear with $s_k = +1$ are labelled by some set of σ integers chosen from $1 \dots n$; this set is denoted by ϱ . The symbol $\varepsilon_\varrho^\sigma = +1$ or -1 , depending on whether the set ϱ is an even or odd permutation of the natural order.

The commutation relations for the Γ_ϱ^σ may be found as follows:

$$\Gamma_\varrho^\sigma \Gamma_{\varrho'}^{\sigma'} = \varepsilon_\varrho^\sigma \varepsilon_{\varrho'}^{\sigma'} (\gamma_1^{s_1} \dots \gamma_n^{s_n}) (\gamma_1^{s'_1} \dots \gamma_n^{s'_n}) = (-)^{s'_1(\sigma - s_1)} (-)^{s'_2(\sigma - s_2)} \dots \Gamma_{\varrho'}^{\sigma'} \Gamma_\varrho^\sigma.$$

Hence

$$(10) \quad \Gamma_\varrho^\sigma \Gamma_{\varrho'}^{\sigma'} = (-)^{\sigma\sigma' + \mathbf{s} \cdot \mathbf{s}'} \Gamma_{\varrho'}^{\sigma'} \Gamma_\varrho^\sigma$$

where

$$(10a) \quad \mathbf{s} \cdot \mathbf{s}' = \sum_1^n s_k s_{k'}.$$

This will be abbreviated:

$$(10b) \quad \Gamma \Gamma' = \mu \Gamma' \Gamma$$

$$\mu = (-)^{\sigma\sigma' + \mathbf{s} \cdot \mathbf{s}'}.$$

Similarly one finds by (6b) and (10b)

$$(11) \quad \begin{cases} EE' = (C^{-1}\Gamma)(C^{-1}\Gamma') = (C^{-1}\Gamma)(\Gamma' C^{-1})\lambda' \\ \quad = \lambda\mu\lambda' E'E \end{cases}$$

where

$$\lambda\mu\lambda' = (-)^{(\sigma+\sigma')p+t+t'} (-)^{\sigma\sigma' + \mathbf{s} \cdot \mathbf{s}'}.$$

Notice also that

$$\Gamma^2 = \varepsilon(\sigma) \Gamma \Gamma^\dagger = \varepsilon(\sigma).$$

In particular

$$C^2 = C^{-2} = \varepsilon(p).$$

Hence

$$E^2 = C^{-1} \Gamma \cdot C^{-1} \Gamma = \lambda C^{-1} \Gamma^2 C^{-1}$$

or

$$(12) \quad E^2 = \lambda \varepsilon(\sigma) \varepsilon(p).$$

3. - Tensors and Invariants.

Define the rotation matrix $S_{\mu\nu}$ by

$$(13) \quad S_{\mu\nu} = 1 + \frac{1}{2} \gamma_\mu \gamma_\nu \omega_{\mu\nu},$$

where $\omega_{\mu\nu}$ is an infinitesimal rotation in the $\mu\nu$ -plane. The induced spinor transformation is

$$(13a) \quad \psi' = S_{\mu\nu} \psi.$$

The inverse transformation is

$$S_{\mu\nu}^{-1} = 1 - \frac{1}{2} \gamma_\mu \gamma_\nu \omega_{\mu\nu}.$$

One has

$$\begin{aligned} C^{-1} S_{\mu\nu}^{-1} C &= 1 - \frac{1}{2} \tilde{\gamma}_\mu \tilde{\gamma}_\nu \omega_{\mu\nu} = 1 + \frac{1}{2} \tilde{\gamma}_{\mu\nu} \omega_{\mu\nu} \\ &= \tilde{S}_{\mu\nu}. \end{aligned}$$

Therefore

$$(14) \quad C^{-1} S^{-1} C = \tilde{S}.$$

Next construct $\tilde{\psi} E \chi$ where ψ and χ are any two spinors. This bilinear expression transforms as follows:

$$\tilde{\psi}' E \chi' = \tilde{\psi} (\tilde{S} E S) \chi.$$

If E is chosen as C^{-1} , then one sees that $\tilde{\psi} C^{-1} \chi$ is an invariant in virtue of (14), and that a general tensor of rank σ may be obtained by forming

$$(15a) \quad A_\sigma^\sigma = \tilde{\psi} E_\sigma^\sigma \chi,$$

where

$$(15b) \quad E_{\varrho}^{\sigma} = C^{-1} \Gamma_{\varrho}^{\sigma}$$

and $\Gamma_{\varrho}^{\sigma}$ is defined in equation (9). The A_{ϱ}^{σ} are antisymmetric in all indices, and there are $n+1$ such tensors altogether.

From them one may construct the $n+1$ « β -invariants»:

$$(16a) \quad F_{\sigma} = \frac{1}{\sigma!} \sum_{\varrho} (\tilde{a} E_{\varrho}^{\sigma} b) (\tilde{c} E_{\varrho}^{\sigma} d),$$

where a, b, c, d are any four spinors. The most general linear combination is

$$(16b) \quad F = \sum_{\sigma}^{n+1} g_{\sigma} F_{\sigma}.$$

4. — Permutation Symmetries.

If the order of the spinors in F_{σ} is changed, one gets a new invariant PF_{σ} which may then be reexpressed as a linear combination of invariants in the original order:

$$(17) \quad PF_{\sigma}(abcd) = \sum_{\sigma'} F_{\sigma'}(abcd) P_{\sigma'\sigma}.$$

In this way one gets an $n+1$ -dimensional representation of the permutation group on $abcd$. Because of the simple form of $F(abcd)$ the complete representation obtained in this way may be generated from the transposition matrices $\|P_{ab}\|$ and $\|P_{ba}\|$. Further P_{ab} is diagonal, since all the E_{ϱ}^{σ} are either symmetric or antisymmetric. In fact, according to equation (7)

$$(18) \quad P_{ab}(\sigma', \sigma) = \delta(\sigma', \sigma) [(-)^{p\sigma} \varepsilon(\sigma) (-)^p E(p)].$$

To obtain a representation of P_{ba} we may start from

$$(19) \quad \begin{aligned} F_{\sigma}(abcd) &= \sum_{\varrho} (a E_{\varrho}^{\sigma} b) (c E_{\varrho}^{\sigma} d) \\ &= \sum_{\varrho} \sum_{ijkl} (E_{\varrho}^{\sigma})_{ij} (E_{\varrho}^{\sigma})_{kl} a_i b_j c_k d_l. \end{aligned}$$

In order to pair the four spinors in the new order, write

$$(20) \quad (E_{\varrho}^{\sigma})_{ij} (E_{\varrho}^{\sigma})_{kl} = \sum_{\sigma' \varrho'} (E_{\varrho'}^{\sigma'})_{il} (E_{\varrho'}^{\sigma'})_{kj} V_{\varrho' \varrho}^{\sigma' \sigma}.$$

Then

$$F_{\sigma}(abcd) = \sum_{\sigma' \varrho'} (a E_{\varrho'}^{\sigma'} d) (c E_{\varrho'}^{\sigma'} b) V_{\varrho' \varrho}^{\sigma' \sigma}.$$

In this last equation $\sum_{\varrho} V_{\varrho'\varrho}^{\sigma'\sigma}$ must be independent of ϱ' ; otherwise the right hand side would not be an invariant. One may therefore introduce

$$(21) \quad P_{bd}(\sigma', \sigma) = \sum_{\varrho} V_{\varrho'\varrho}^{\sigma'\sigma}.$$

Then

$$(22) \quad F_{\sigma}(abcd) = \sum F_{\sigma'}(abcd) P_{bd}(\sigma', \sigma).$$

Equation (22) therefore gives a representation of P_{bd} . It may be calculated explicitly as follows. From (20) one finds

$$\sum_{ijkl} (E_{\varrho}^{\sigma})_{ij} (E_{\varrho''}^{\sigma''})_{jk} (E_{\varrho}^{\sigma})_{kl} (E_{\varrho'}^{\sigma'})_{li} = \sum_{ijkl} (E_{\varrho'}^{\sigma'})_{il} (E_{\varrho''}^{\sigma''})_{li} (E_{\varrho'}^{\sigma'})_{kj} (E_{\varrho''}^{\sigma''})_{jk} V_{\varrho'\varrho''}^{\sigma'\sigma''}$$

or

$$(23) \quad V_{\varrho''\varrho}^{\sigma''\sigma} = N^{-2} \text{Tr} [E_{\varrho}^{\sigma} E_{\varrho''}^{\sigma''} E_{\varrho}^{\sigma} E_{\varrho''}^{\sigma''}]$$

since the E_{ϱ}^{σ} form an orthonormal basis.

By (11) and (12)

$$(24) \quad \begin{cases} V_{\varrho'\varrho}^{\sigma'\sigma} = N^{-2} \lambda \lambda' \mu \text{Tr} (E_{\varrho'}^{\sigma'})^2 (E_{\varrho}^{\sigma})^2 \\ \quad = N^{-1} (\mu) \varepsilon(\sigma) \varepsilon(\sigma') \\ \quad = N^{-1} (-)^{s \cdot s'} \varepsilon(\sigma + \sigma') \end{cases}$$

and

$$(25) \quad P_{bd}(\sigma', \sigma) = N^{-1} \varepsilon(\sigma + \sigma') w(\sigma', \sigma)$$

where

$$(26) \quad w(\sigma', \sigma) = \sum_s (-)^{s \cdot s'}.$$

Here $s \cdot s'$ is according to (10a) the inner product of two n -dimensional vectors whose components have values restricted to 1 or 0. Since $w(\sigma', \sigma)$ is independent of s' , it is most convenient to make a definite simple choice of s' , namely: $s'_k = 1$, $k = 1, 2, \dots, \sigma'$; $s'_k = 0$, $k = \sigma' + 1, \dots, n$. On the other hand s must be summed over all possibilities consistent with its length, σ . This sum may be found as follows: Suppose that m of the first σ' components of s do not vanish, as shown:

| | | |
|--------|--|--|
| | $\longleftarrow \sigma' \longrightarrow$ | $\longleftarrow n - \sigma' \longrightarrow$ |
| s' : | 1 ... 1 | 0 ... 0 |
| s : | 1 appears m times | 1 appears $\sigma - m$ times |

Then $ss' = m$. There are $C_m^{\sigma'}$ ways of distributing m marks on σ' sites and there are $C_{\sigma-m}^{n-\sigma'}$ ways of distributing the remaining marks on the remaining places. Hence

$$(27) \quad w(\sigma', \sigma) = \sum_0^{\sigma'} C_m^{\sigma'} C_{\sigma-m}^{n-\sigma'} (-)^m.$$

This completes the determination of $P_{bd}(\sigma', \sigma)$. All other matrices in this representation may be generated from P_{ab} and P_{bd} . For example

$$P_{ad} = P_{ab} P_{bd} P_{ab}$$

or

$$(28) \quad \begin{aligned} P_{ad}(\sigma', \sigma) &= (-)^{\mathfrak{p}(\sigma+\sigma')} \varepsilon(\sigma) \varepsilon(\sigma') P_{bd}(\sigma', \sigma) \\ &= N^{-1} (-)^{\mathfrak{p}(\sigma+\sigma')} (-)^{\sigma\sigma'} w(\sigma', \sigma). \end{aligned}$$

5. - Application.

In the physically interesting case n is 6, the matrices are eight-rowed, and the wave functions have eight components. A particle is therefore characterized by an additional dichotomic variable, τ , which one may be tempted to associate with isotopic spin. There are seven invariants instead of the usual five, and these may be classified by changes in τ as well as in the usual angular momentum; if τ does not change there are fewer than five invariants.

Let the interaction between four fermion fields be $F(a, b, c, d)$. If these mutually anticommute, and if the number of lines leading into a vertex is equal to the number leading out, the part of F (say F') contributing to transition probabilities is antisymmetric in a and c and in b and d :

$$(29) \quad P_{ba} F'(abcd) = -F'(abcd).$$

Let P be any permutation on $abcd$. As previously noted one gets a seven dimensional representation of the permutation group on $abcd$ according to the equation

$$P F_{\sigma}(abcd) = \sum_{\sigma'} F_{\sigma'}(abcd) P(\sigma', \sigma),$$

where the F_{σ} are the seven invariants. The complete interaction may be regarded as a vector with components g_{σ} :

$$F = \sum_{\sigma}^7 g_{\sigma} F_{\sigma}.$$

Define \bar{g}_σ by

$$PF = \sum_1^7 \bar{g}_\sigma F_\sigma.$$

Then

$$\bar{g}_{\sigma'} = \sum_1^7 P_{\sigma'\sigma} g_\sigma.$$

It is seen that

$$(30) \quad P_{ad}^{-1} P_{bd} P_{ad} = P_{ab}.$$

Since P_{ab} is diagonal, the eigenvectors of P_{bd} are the columns of P_{ad} . Those corresponding to the eigenvalue -1 and therefore satisfying (29) are the analogues of the more familiar forms $S-T+P$, $V-A$, and $S-A-P$.

The possibility of attaching physical significance to these forms depends on an assignment of τ to the different fermions in such a way that the observed beta couplings can be realized; the so called universal Fermi coupling would then be the same only in terms of the eight rowed spinors.

RIASSUNTO (*)

Si propone un'equazione d'onda covariante nelle trasformazioni conformi. Partendo dalle funzioni d'onda a otto righe si costruiscono invarianti beta generalizzati che soddisfano questa equazione e si derivano alcune altre proprietà formali.

(*) Traduzione a cura della Redazione.

On Non-local Form Factors.

R. J. FINKELSTEIN

Department of Physics, University of California - Los Angeles

(ricevuto il 21 Marzo 1955)

Summary. — A non-local formalism is suggested for relating classical ideas about unitary fields with the current methods of making calculations about elementary particles. In the proposed scheme the form factors and masses of all particles are obtained from a single invariant through a classical lagrangian principle.

1. — Introduction.

It is possible to give a simple covariant description of extended particles in a classical, nonlinear field theory. Although this simplicity does not survive quantization, the complications result from formal rules which are very indirectly related to experiment. As a consequence it is not known at the present time whether the simple concepts of a classical unitary theory are in unavoidable conflict with experiment.

The following properties of such a theory may be recalled. Elementary particles are regarded as non-singular concentrations of field and correspond to classical eigensolutions of the field equations; as a consequence their masses are calculable and discrete. It is quite possible in principle to describe all particles of spin $\hbar/2$ by the different eigensolutions of the same classical eigenproblem ⁽¹⁾.

The fundamental difficulties in this procedure become obvious when one asks a typical question: for example, about the decay of one particle into several others, or about the collision of two of them with or without multiple production. Here the mathematical difficulties exclude any expectation of a

⁽¹⁾ R. FINKELSTEIN, R. LELEVIER and M. RUDERMAN: *Phys. Rev.*, **83**, 326 (1951); R. FINKELSTEIN, P. KAUS, to appear.

rigorous solution. Moreover, even though the field equations do in principle answer these questions, one is faced with an extreme case of the mathematical situation made familiar by kinetic theory: even if the differential equations could be solved, one still would have to resort to statistical methods because the boundary conditions are not controllable. The field structure representing a *single* particle is analogous to a gas — but with a continuously infinite instead of only a finite number of degrees of freedom. The representative point of a gas or particle describes an almost unrestricted motion in phase space; in the former case only the temperature, pressure, and volume, may be known and in the latter, only the charge, mass, and spin. The particle as well as the gas may be represented by a Gibbs ensemble. It is clear that any realistic classical theory of the elementary particles would rest on statistical foundations.

Indeed, it is possible, according to a well known conjecture ⁽²⁾, that quantal formulas are statistically abbreviated ways of describing classical interactions between the classical field structures which represent the elementary particles. Here this view will be formulated in the following statements:

- a) the structure of an isolated particle is determined by classical field equations;
- b) the observable interactions of the elementary particles with each other and with macroscopic objects are described by quantal equations.

If the classical theory is given, then it is possible in principle to establish the truth or falsity of b), and ultimately this would have to be done. But there may be a simpler tentative approach which attempts only to relate, through a set of mutually consistent postulates, classical ideas about unitary fields with the usual methods of doing calculations about elementary particles.

If a) and b) were in fact correct, then one might expect that the empirical masses (and other self properties) which appear in the usual quantal equations are at least approximately the same as the masses which characterize isolated classical particles. Furthermore one might also expect that the classical eigen-solutions play an important role in the quantal equations. For example if an isolated particle decays, then according to the picture underlying a) and b) the quantal description of the decay must somehow depend upon the classical eigensolutions which characterize the initial and final particles. In this spirit one may then try to find the simplest formal way of introducing these classical eigenfunctions into quantum field theory without getting into any new experimental conflicts. We have attempted to do this with a non-local formalism which may have the merit of being suggestive.

⁽²⁾ This point of view has been discussed by EINSTEIN, DE BROGLIE, BOHM and others. See, for example, L. DE BROGLIE: *Nuovo Cimento*, **1**, 37 (1955).

2. - Formulation.

For definiteness suppose that the elementary particles are fermions, that their only interaction is the so-called universal interaction, and that the bosons are composite. Associate with any state of an elementary particle a non-local operator $\psi_p(x_1, x_2)$ depending on two points of space x_1, x_2 and an energy momentum vector p (which designates the rest mass as well as the state of motion). Postulate the following:

1) ψ_p may be factored

$$(A) \quad \psi_p(x_1, x_2) = V_p(\bar{x}) X_p(x) a_p$$

where

$$\begin{aligned} \bar{x} &= \frac{1}{2}(x_1 + x_2) & p^2 &= m_p^2 \\ x &= x_2 - x_1 & [\bar{a}_p, a_q]_+ &= \delta_{pq}, \text{ etc.} \end{aligned}$$

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_p \right) V_p = 0,$$

$V_p(\bar{x})a_p$ is the usual product of an eigenstate with an absorption operator, and $X_p(x)$ is to be understood as a form factor depending on M . The operator, ψ_p , associated directly with a particle state, and not $\sum \psi_p$ is to be regarded as the primary physical entity. If $X_p(x)$ is not a δ -function, the theory is non-local.

2) We propose to determine the $X_p(x)$ from a classical lagrangian principle

$$(1) \quad \frac{\delta S}{\delta \bar{X}_p} = 0,$$

where S , which depends only on commuting variables, is invariant under the complete physical group. This condition is motivated by our earlier assumption a) that the structure of an isolated particle is determined by classical field equations. Equation (1) may be regarded as set of non-linear field equations and the X_p may be thought of intuitively as their particle-like solutions. Since S is by hypothesis invariant under the complete gauge and Lorentz groups, there exist the usual classical integrals of charge, energy-momentum, and angular momentum. For example, the charge is

$$(2) \quad q = e \int \bar{X}_p \gamma_4 X_p dx,$$

in the spinor case. The energy momentum is

$$(3) \quad p_\mu = \sum_\alpha \left(\frac{\partial S}{\partial \dot{X}_\alpha} \right) \partial_\mu X_\alpha - S \delta_{4\mu}.$$

One of course requires that these integrals be finite, and that at once leads to an eigenproblem for the X_p . The various eigensolutions X_p then define the elementary particles in this theory.

So long as form factors remain arbitrary, non local theories can not be tested. Equation (1) has the advantage that in removing this arbitrariness it automatically leads to self-properties and a discrete mass spectrum. These self-properties, mass, charge, and spin, result from the postulated invariance of S to the complete physical group. It is important that although these integrals are extended over the internal space, they have the correct transformation properties in the external space, since $x = x_2 - x_1$ and $\bar{x} = \frac{1}{2}(x_2 + x_1)$ transform in the same way. Intuitively this last remark means for example, that a change in velocity of the particle as a whole leads to a Lorentz contraction in its shape. The X_p will be referred to as structure functions.

3. - Quantum Observables.

Equation (1) and the associated tensors like p_μ and q_p are classical and refer to isolated particles. In terms of them define the charge and energy momentum operators.

$$(4) \quad Q = \sum q_p (\bar{a}_p a_p - a_p \bar{a}_p),$$

$$(5) \quad P_\mu = \sum p_\mu (\bar{a}_p a_p - a_p \bar{a}_p).$$

Let

$$(6) \quad \psi_{r...q} \dots = \bar{a}_r \dots a_q \dots \psi_0$$

be the wave function of a set of positive and negative charges, where ψ_0 is the vacuum wave function. Then

$$(7) \quad Q \psi_{r...q} \dots = \sum q_p (N_p^+ - N_p^-) \psi_{r...q} \dots = e(N^+ - N^-) \psi_{r...q} \dots,$$

where N^+ and N^- represent the number of positive and negative charges and $e = q_p$. The integral q_p is a scalar and does not depend on the state of motion but it might depend on the rest mass. In replacing q_p by e , it has been assumed for simplicity that all particles have the same charge. Then in the

spinor case

$$(8) \quad q_p = \varepsilon \int \bar{X}_p \gamma_4 X_p dx = e ,$$

where ε is the coupling constant, as usual taken to be $e/\hbar c$. This condition fixes the normalization of X_p . Since the integral q_p does not depend on p , the normalization may be carried out in any frame, most conveniently in the proper frame. Conditions (1) and (8) define the X_p . The momentum p_μ , associated with any state X_p may be determined from the internal energy momentum tensor (3), and the external factor $V_p(\bar{x}) = U_p \exp[p_\mu \bar{x}_\mu]$ is fixed as soon as p_μ is specified.

The special solutions $\bar{X}(x) \exp[i\omega t]$ are most easily normalized since they carry no momentum and therefore represent particles at rest in the given frame. Structure functions of moving particles may be generated by Lorentz transformation of these special solutions.

4. - Interactions.

One can perform the customary calculations as soon as the S or η matrix is specified:

$$(9) \quad \begin{cases} \bar{\Psi}(\infty) = S\Psi(-\infty) \\ S = e^{i\eta} , \end{cases}$$

η is usually taken to be the integral of the interaction lagrangian over space-time

$$\eta = \int L^{\text{int}} d^4x .$$

The interaction lagrangians currently used depend upon certain parameters (charge, mass, etc.) which commute with all fields. In local theories with non-local interactions (which are equivalent to non-local theories) they contain certain form factors, supposedly universal functions which also commute with all fields. Let us refer to such a commuting spacetime function in the lagrangian as a classical or c -field. In the model here proposed these c -fields are the solutions of (1) and (8), the structure functions. The method of introducing these functions into η is clearly quite arbitrary until the relation between the

basic postulates *a*) and *b*) is clarified. Following YUKAWA ⁽³⁾ one might write

$$\eta = g \iiint \bar{\psi}(x_1, x_2) \dots \psi(x_2, x_3) \bar{\psi}(x_3, x_4) \dots \psi(x_4, x_1) dx_1 \dots dx_4.$$

A simpler possibility is perhaps

$$(10) \quad \eta = g \int \int [\bar{\psi}_r(\bar{x}, x) \Gamma \psi_s(\bar{x}, x)] \cdot [\bar{\psi}_t(\bar{x}, x) \Gamma \psi_u(\bar{x}, x)] d^4 \bar{x} d^4 x,$$

where Γ is an outer product of matrices belonging to internal and external spaces. Then, except for emission and absorption operators,

$$\eta = g_{rstu} \int [\bar{V}_r(\bar{x}) \Gamma_{\bar{x}} V_s(\bar{x})] \cdot [\bar{V}_t(\bar{x}) \Gamma_{\bar{x}} V_u(\bar{x})] d^4 \bar{x},$$

$$g_{rstu} = g \int [\bar{X}_r(x) \Gamma_x X_s(x)] \cdot [\bar{X}_t(x) \Gamma_x X_u(x)] d^4 x.$$

This formulation of the hypothesis of a universal interaction introduces a single Fermi constant g but leads to an effective g_{rstu} which depends on the overlap of the structure functions of the four particles. In principle therefore the single invariant S determines not only the masses of all the particles but also their interactions, which will in general depend upon internal quantum numbers characterizing the X .

In the preceding discussion it was assumed for definiteness that the elementary particles are spinors, and that the bosons are composite. If this assumption is not made, then one would have to understand that the irreducible fields described by (1) have components transforming like integral as well as half-integral representations of the Lorentz group. The discussion would then be altered by postulating instead of (10) say,

$$(11) \quad \eta = g \int \int [\psi_e(\bar{x}, x) \Gamma] \cdot [\bar{\psi}_t(\bar{x}, x) \Gamma \psi_u(\bar{x}, x)] d^4 \bar{x} d^4 x,$$

or, ignoring the a_v ,

$$\eta = g_{etu} \int [V_e(\bar{x}) \Gamma_{\bar{x}}] \cdot [\bar{V}_t(\bar{x}) \Gamma_{\bar{x}} V_u(\bar{x})] d^4 \bar{x},$$

⁽³⁾ A non-local beta theory of the Yukawa type was proposed by O. HARA, T. MARUMORI, Y. OHNUKI and H. SHIMODAIRA: *Nuovo Cimento*, **12**, 309 (1954).

where

$$g_{etu} = \int [X_e(x) \Gamma_x] \cdot [\bar{X}_t(x) \Gamma_x X_u(x)] d^4x.$$

A meson theory of this type exhibits a covariant cutoff. Again the single invariant S fixes, in a unique way, not only the masses of all the particles but also the covariant cutoffs.

The crucial point remains the relation between postulates $a)$ and $b)$. If the conjecture which they express is correct, progress depends on the construction of a moderately realistic classical theory; for according to the possibility discussed in this note, the characteristic solutions and masses appearing in the classical theory have a meaning in the complete theory.

RIASSUNTO (*)

Si suggerisce un formalismo non locale per mettere in relazione le concezioni classiche sui campi unitari con i metodi in uso per l'esecuzione dei calcoli concernenti le particelle elementari. Nello schema proposto i fattori di forma e le masse di tutte le particelle si ottengono da un'unica invariante applicando un classico principio lagrangiano.

(*) Traduzione a cura della Redazione.

Spurious Peaks due to Multiple Backscattering in a Single Crystal Gamma Spectrometer.

A. BISI, and L. ZAPPA

Istituto di Fisica Sperimentale del Politecnico - Milano

E. GERMAGNOLI

Laboratori CISE - Milano

(ricevuto il 22 Marzo 1955)

Summary. — The energy spectra obtained with a single crystal spectrometer for γ -rays have been investigated in detail in the energy range between 50 and 300 keV. Several γ sources have been used. Evidence is given of some effects due to backscattering of γ -rays from the materials surrounding the crystal. The influence of this effects upon the reliability of the spectrometer in the considered interval of energy is discussed.

1. — Introduction.

When single crystal scintillation detectors are used to study the spectrum of γ -rays emitted from a source in the energy range between 60 keV and 250 keV approximately, a remarkable limitation can take place due to the fact that some spurious peaks come out whenever the investigated source emits also harder γ -radiations. Such peaks, which are sufficiently well defined in energy, are due to Compton scattering processes which primary γ -rays undergo within the materials that surround the phototube and within the phototube itself. This fact is well known and usually taken into account as far as the most prominent peak is concerned. The origin of this peak has been often discussed in the literature where it is attributed to scattering processes of γ -rays with scattering angles not far from 180° , so that the scattered quantum is able to strike the crystal again ⁽¹⁾.

⁽¹⁾ K. LIDÉN and N. STARFELT: *Ark. för Fysik*, **7**, 427 (1954).

In a recent paper ⁽²⁾ this phenomenon has been investigated in detail and evidence has been found of a second, less energetic peak. However, as the geometrical arrangement used in the above mentioned investigation is not usual in works with scintillation spectrometer and has been chosen for the purpose of magnifying the effect, it seemed worthwhile to us to examine this problem again with an experimental arrangement which can be considered as most common in this kind of measurements and with very weak sources.

The present work has the main purpose of avoiding the danger of exhibiting wrong decay schemes in which the peaks due to multiple scattering are interpreted as due to weak γ transitions. Actually it is easy to find in the literature some cases in which low energy γ -rays are questionable, owing to the fact that their energies and intensities are practically coincident with the energies and the intensities attributable to harder γ -rays which have undergone one or several large angle Compton scatterings.

2. - Experimental Results.

Fig. 1 shows the experimental arrangement. Materials which could enhance the effect have been carefully avoided and the Al containers of the crystal and of the phototube are very thin. Only in some measurements had we to interpose a suitable layer of polythene between source and crystal in order to shield the crystal from β -particles emitted from the source. The effect of a brass scatterer has been also investigated.

The amplification chain is a conventional one and the spectral distribution of γ -rays has been analyzed by means of a fast, twenty channel pulse analyzer ⁽³⁾.

Sources of ^{137}Cs (662 keV), ^{60}Co (1.17 and 1.31 MeV), ^{88}Sr (513 keV) and ^{64}Cu (positron emitter) have been used and the energy interval between 60 keV and 300 keV has been carefully investigated in each case.

In order to investigate a possibly different behaviour of the two most commonly used phototubes, measurements have been carried out both with Dumont 6292 and with RCA 5819 phototubes, but remarkable differences in the spectra were not evident. The influence of the sizes of NaI(Tl) crystals has also been tested, but the small differences in the shapes

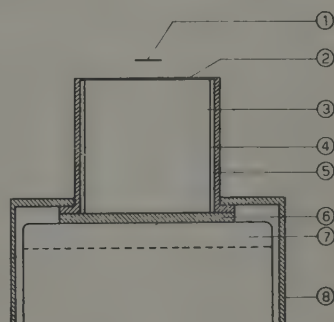


Fig. 1. - Experimental arrangement. (1) Source; (2) very thin Al sheet; (3) NaI(Tl) crystal; (4) MgO; (5) Al; (6) glass; (7) photocathode; (8) Al.

⁽³⁾ E. GATTI: *Nuovo Cimento*, 11, 153 (1954).

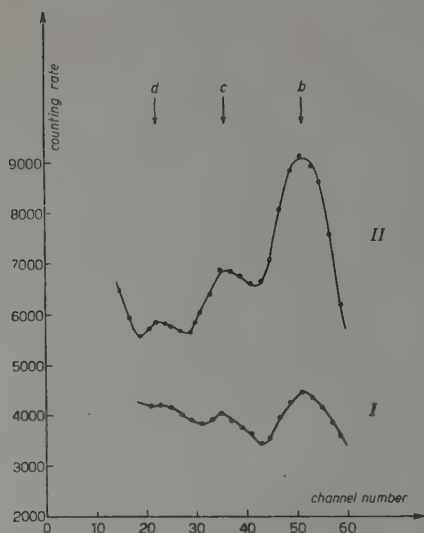


Fig. 2. — Peaks due to backscattering with a source of ^{85}Sr : curve I, no backscatterer added; curve II, brass backscatterer added.

of the spectra which have been found are not significant as they can be attributed to the dependence of the resolution upon the size of the crystal.

Fig. 2 shows the results we obtained with a source of ^{85}Sr : curve I refers to the experimental arrangement given in Fig. 1; curve II has been obtained when crystal and γ source were surrounded with a brass scatterer about 2 cm thick; the difference in the counting rates and also in the relative intensities of the backscattering peaks are fairly evident when the two curves are compared.

A similar curve has been obtained with a source of ^{60}Co and is given in Fig. 3. The differences between the energies of the peaks and their meanings will be discussed in the next section.

A full account of the energy spectrum obtained with the source of ^{85}Sr is given in Fig. 4. Besides the Compton edge of the 513 keV line, four peaks of different intensities can be observed. They are due to backscattering phenomena and are quoted with the letters: *d*, *c*, *b*, *a*.

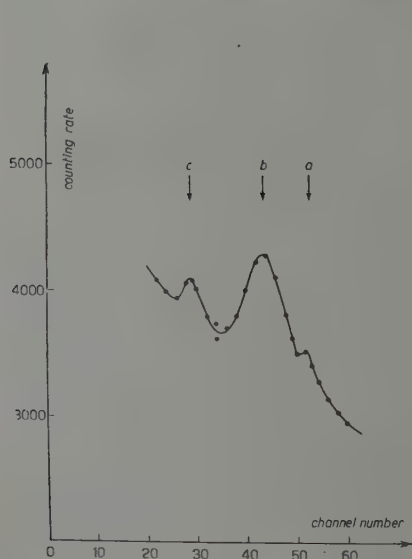


Fig. 3. — Peaks due to backscattering with a source of ^{60}Co . No backscatterer.

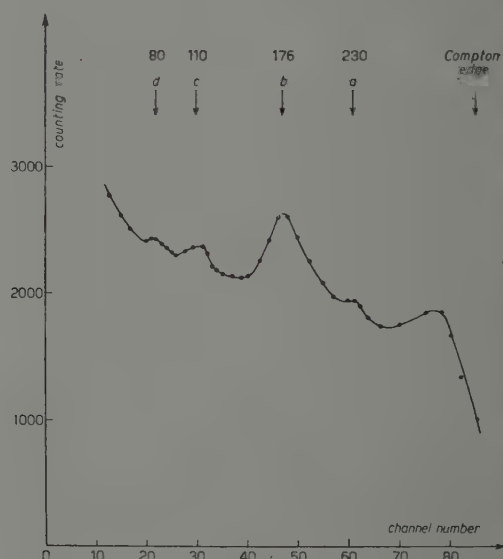


Fig. 4. — Peaks due to backscattering with a source of ^{85}Sr . No backscatterer.

3. - Discussion.

The interpretation of the peaks given in the preceding figures can be easily made if the well known calculation concerning the energy of the most prominent backscattering peak is repeated and extended. Such interpretation can be summed up as follows.

We are mainly interested here in the behaviour of the quanta which do not strike directly the crystal, or which cross the crystal without any loss of energy within it; it is possible that such γ -rays, after one or more scattering processes, cross again the crystal. Their energy is in this case $E' < E$, where E is the primary energy. The probability that such γ -rays are absorbed to give photoelectric effect within the crystal is greater when E' is smaller, and in any case much greater than for primary γ -rays.

The most likely case is that of a quantum scattered by materials surrounding the crystal within an angle between 120° and 180° , which then strikes the crystal with the energy:

$$E' = \frac{E}{\frac{E(1 - \cos \theta)}{mc^2} + 1},$$

that it loses in striking. Here θ is the scattering angle and mc^2 is the rest energy of the electron. To such phenomenon, as it is well known, the peak quoted as *b* in Fig. 4 can be attributed. Peak *a* is remarkably less intense but quite evident in any case. It can be explained in the manner indicated in Fig. 5.

γ -rays which have already been scattered, with a loss of energy given by the above formula, can undergo further scattering processes with scattering angles always larger than 90° . It is probably superfluous to make a calculation which is moreover obvious at least in principle. We restrict ourselves to pointing out that γ -rays striking the crystal after a rather small number of collisions have an energy distribution which is mainly a function of the number of the collisions undergone. The maximum of this distribution is not critically

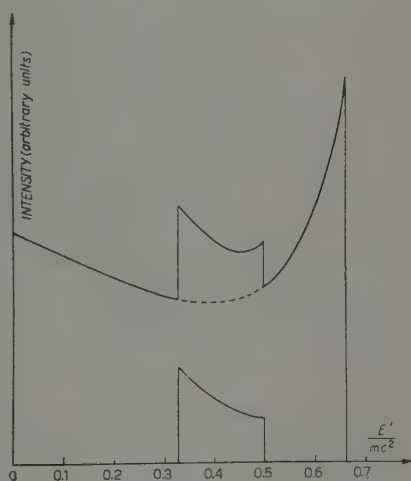


Fig. 5. - Lower curve: ideal energy distribution of the scattered quanta in the materials surrounding the crystal ($90^\circ \leq \theta \leq 180^\circ$). Upper curve: ideal energy distribution resulting from all scattered quanta both within and outside the crystal. Energy of the incident quantum: $E = mc^2$.

dependent either upon the actual values of the scattering angles, or the order with which the collisions take place, at least if every angle of scattering is somewhat larger than 120° .

TABLE I.

| E (keV) | E_b | | E_c | | E_d | | E_e | |
|--------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| | 120° | 180° | 120° | 180° | 120° | 180° | 120° | 180° |
| 511 | 205 | 170 | 128 | 102 | 93 | 73 | 73 | 57 |
| 1240 | 268 | 213 | 151 | 116 | 105 | 80 | 80 | 61 |

TABLE II.

| E (keV) | E_b | E_c | E_d | E_e |
|--------------|-------|-------|-------|-------|
| 511 | 176 | 110 | 70 | — |
| 1240 | 220 | 133 | — | — |

In order to illustrate our assertion we give in Table I the energies of scattered γ -rays which result from two hypotheses, one that every collision takes place with a scattering angle equal to 120° and another that the angle is equal to 180° . It appears that the energy of the scattered γ -rays is not critically dependent upon the angle of scattering, in the considered range of angles. The calculation has been done for the primary energies of 511 keV, corresponding to the source of ^{85}Sr , and of 1.24 MeV which represents an average value of the energies of the two γ lines emitted from ^{60}Co .

Obviously the experimental values have to be included within these two extreme values and it seems reasonable to expect that the corresponding peaks should be both broader and less intense when the number of collisions undergone by γ -rays is rather high. The agreement between the experimental values and the calculated ones is satisfactory, particularly under the hypothesis that every angle of scattering is quite near to 180° : this fact is shown in Table II. Similar results have been obtained with the sources of ^{137}Cs and ^{64}Cu .

It is also interesting to point out that the peaks b , c , d , in Fig. 4 have always one side which is definitely steeper than the other one, alternatively towards the high and the low energy region. This fact is in agreement with the well known threshold for the energy of the scattered quantum: actually no γ -ray can be backscattered ($\theta = 180^\circ$) with an energy lower than:

$$E^* = \frac{E}{(2E/mc^2) + 1},$$

The shape of the peak *b* in Fig. 4 can be accounted for in this way and the same reasoning holds true, with obvious modifications, for the other scattering peaks.

4. - Conclusions.

The above described results show that it is substantially impossible to get rid completely of the spurious peaks due to backscattering, as at least part of the backscattered intensity has to be ascribed to the materials with which the phototube is made. An attempt of minimizing this effect has been done by means of surrounding the crystal with a lead absorber and collimating the impinging beam of γ -rays with lead also. The total intensity of the backscattering peaks turned out to be remarkably reduced, but the intensity ratios between the individual lines did not noticeably change; this fact agrees with the hypothesis that the contribution to backscattering by the materials located below the crystal (dynodes and photocathode) is appreciable.

A disadvantage brought by the lead collimator, or in fact by any other material of high *Z* used for collimation, is the excitation of the characteristic fluorescence radiation of the material. X-rays emitted from lead can be largely absorbed with thin layers of suitable elements which surround the crystal though they cannot be completely eliminated.

We wish to thank Prof. G. BOLLA for his kind interest in the present work.

RIASSUNTO

Gli spettri di energia ottenuti con uno spettrometro a scintillazione per raggi γ sono stati esaminati in dettaglio nella regione di energia compresa tra 50 e 300 keV usando differenti sorgenti di raggi γ . Si sono messi in evidenza effetti dovuti a diffusione dei γ da parte dei materiali circostanti il cristallo. Si discute la loro influenza sulla possibilità di analisi di spettri γ a bassa energia.

Angular Distribution in Cosmic Ray Stars.

E. LOHRMANN

*Hochspannungslaboratorium Hechingen
des Max Planck-Instituts für Physik der Stratosphäre*

(ricevuto il 28 Marzo 1955)

Summary. — The angular distribution of star producing particles and their secondaries was measured under absorbers of Fe, Sn and C at altitudes of 2 960 m (Zugspitze) and 420 m (Weissenau). The angular distribution of ionizing star producing particles can approximately be represented by a $\cos^6 \theta$ law at 2 960 m and by $\cos^{5,7} \theta$ at 420 m. With increasing energy of the particles the distribution appears to be stronger collimated towards the vertical. The angular distribution of grey tracks around the direction of the star producing particle agrees with results of the Bristol Group for stratosphere altitudes, suggesting the distribution to be independent of the energy of the star producing particle. With increasing energy of the secondaries an increasing collimation of the angular distribution is observed. The angular distribution of grey tracks relative to the vertical is given for neutron- and for proton-induced stars respectively. An attempt was made to derive the angular distribution of the neutral star producing component leading to a $\cos^n \theta$ law with $1 < n < 4$.

1. — Introduction.

This work was carried out as contribution to the investigations of our laboratory concerning the transition effects of the star producing components of the cosmic radiation. These transition effects, observed in solid absorbers of Fe, Sn and C required a closer examination of the behaviour of

the nucleonic component in those absorbers (see RÖSSLE and SCHOPPER) ⁽¹⁻⁸⁾.

We have investigated the angular distributions of star producing particles and of grey tracks.

For these investigations we used Ilford G 5 nuclear research plates (area 5×5 cm, emulsion thickness 500 μ), which had been exposed under the mentioned solid absorbers of Fe, Sn and C on the top of the Zugspitze (2960 m) and at Weissenau (420 m).

The tracks emerging from every star (σ -stars excluded) were classified according to their grain-density g :

1) $g < 1.5 g_{\min}$, energy $E > 350$ MeV for protons.

The track is regarded as due to a star producing particle (« primary »), if it is coming from the upper hemisphere; among several such particles we chose the one with the smallest grain-density, or the one enclosing the smallest angle with the vertical.

2) $1.5 g_{\min} < g < 2.7 g_{\min}$, energy $E: 350 \text{ MeV} > E > 140 \text{ MeV}$ for protons.

3) $2.7 g_{\min} < g < 4 g_{\min}$, energy $E: 140 \text{ MeV} > E > 70 \text{ MeV}$ for protons.

Most of these grey tracks are due to protons recoiling from fast nucleons (« grey particles »).

We measured the number $H(\vartheta)d\vartheta$ of tracks in each group enclosing an angle between ϑ and $\vartheta + d\vartheta$ with the vertical. $\vartheta = 0$ means a track going upwards. $P(\vartheta)d\Omega$ is the number of tracks going into an element of solid angle $d\Omega$ with a polar angle ϑ . We have

$$(1) \quad H(\vartheta) = 2\pi P(\vartheta) \sin \vartheta.$$

2. — Influence of dip.

The measured number of tracks $H(\vartheta)$ having a polar angle ϑ was corrected for losses due to tracks dipping steeply in the emulsion. Such tracks cannot be classified according to their grain-density (sec. 1), if it becomes too high to be

(1) E. SCHOPPER, K. H. HÖCKER and G. KUHN: *Phys. Rev.*, **82**, 445 (1951).

(2) E. SCHOPPER, K. H. HÖCKER and G. KUHN: *Phys. Rev.*, **82**, 446 (1951).

(3) E. SCHOPPER, K. H. HÖCKER and E. RÖSSLE: *Zeits. Naturfor.*, **6a**, 603 (1951).

(4) E. RÖSSLE: *Diplomarbeit Stuttgart* (1952).

(5) E. RÖSSLE and E. SCHOPPER: *Zeits. Naturf.*, **9a**, 836 (1954).

(6) G. KUHN and E. SCHOPPER: *Zeits. Naturf.*, **9a**, 851 (1954).

(7) E. RÖSSLE: *Dissertation Stuttgart* (1954).

(8) G. KUHN: *Dissertation Stuttgart* (1954).

reliably measured. We have calculated the expected fraction of grey tracks found making an angle α with the emulsion, assuming that all tracks having an apparent grain-density $g/\cos \alpha > 4g_{\min}$ are lost (full line in Fig. 1). It is

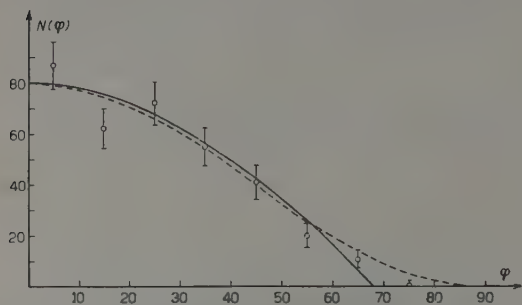


Fig. 1. — \bigcirc number of grey tracks in an interval of 10° of the azimuthal angle φ ; — expected fraction of tracks; --- approximation eq. (2).

to be compared with the number of grey tracks found at a given azimuthal angle φ and having a polar angle $\vartheta = 90^\circ \pm 10^\circ$, thus $\alpha \approx \varphi$. The dotted line in Fig. 1 represents the function $\cos^2 \varphi$, which is used as a suitable approximation in the following calculations.

We have the number of tracks found with an azimuthal angle between φ and $\varphi + d\varphi$:

$$(2) \quad a(\varphi) d\varphi = a(0) \cos^2 \varphi d\varphi.$$

$2\pi a(0)$ is the number of tracks actually present in the plates. For arbitrary angles ϑ we get for the number $H(\vartheta)$ of measured tracks:

$$(3) \quad \bar{H}(\vartheta) = C \int_{\varphi=0}^{\pi/2} a(\alpha) d\varphi.$$

The number of tracks actually present is

$$(4) \quad H(\vartheta) = C \int_{\varphi=0}^{\pi/2} a(0) d\varphi = \frac{\pi \cdot C}{2} a(0).$$

The integral in equ. (3) is evaluated by noting

$$(5) \quad \sin \alpha = \sin \varphi \sin \vartheta.$$

We get from equ. (2), (3), (4), (5):

$$(6) \quad H(\vartheta) = \frac{\bar{H}(\vartheta)}{1 - \frac{1}{2} \sin^2 \vartheta}.$$

The angular distributions of grey tracks relative to the vertical were corrected according to equ. (6). The statistical error of $H(\vartheta)$ is

$$(7) \quad \frac{\delta H}{H} = \frac{\delta \bar{H}}{\bar{H}} + \frac{\delta K}{K} \quad \text{with} \quad K(\vartheta) = \frac{H(\vartheta)}{\bar{H}(\vartheta)}$$

and $\delta\bar{H} = \sqrt{\bar{H}}$;

$$(8) \quad \text{for } \frac{\delta K}{K} \quad \text{we choose} \quad \frac{\delta K}{K} = \frac{K-1}{10}$$

as δK must be 0 for $K=1$; for $K=2$ ($\vartheta = 90^\circ$) $\delta K/K = 10\%$ is derived from the statistical accuracy of Fig. 1.

The angular distributions of star producing particles of energy $E > 350$ MeV were not corrected in this way; a similar calculation showing that this correction does not influence the distributions for angles $\vartheta < 70^\circ$. For $\vartheta > 70^\circ$ it would affect the distributions within the limits of the statistical errors.

3. - Angular Distributions of Ionizing Star Producing Particles at an Altitude of 2960 m.

The angular distribution of ionizing star producing particles having an energy $E > 350$ MeV is shown in Fig. 2; the definition of a star producing particle is given in sec. 1.

We must estimate the error introduced by shower particles running into the upper hemisphere which therefore are regarded as star producing particles. As these shower particles are mostly emitted into the direction of the primary particle, we may assume most of the particles producing such stars to come from the lower hemisphere. We estimate the fraction of star producing neutrons coming from the lower hemisphere to be 15% (see sec. 7). The fraction of ionizing star producing particles coming from the lower hemisphere is 10%.

Only 50% of all these particles

have an energy $E > 1$ GeV sufficient to produce shower particles, as can be deduced from a publication by DILWORTH and GOLDSACK⁽⁹⁾ on the distribution in energy of ionizing star producing particles at an altitude of 3000 m,

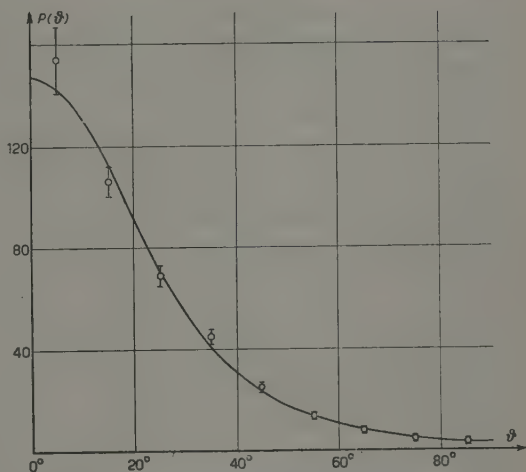


Fig. 2. — \bar{Q} distribution of polar angles of ionizing star producing particles, $E > 350$ MeV at an altitude of 2960 m; $P(\vartheta)$ arbitrary units; — approximation eq. (9).

(9) C. C. DILWORTH and S. J. GOLDSACK: *Nuovo Cimento*, **10**, 926 (1953).

and also from the statistics of our own plates. Estimating the number of neutrons to 1.2 times the number of protons for an average energy of 2 GeV according to CLEMENTEL and PUPPI^(10,11), the fraction of particles which are erroneously regarded as star producing is $\frac{1}{2}(0.1 + 1.2 \cdot 0.15) = 14\%$. As the shower particles are emitted almost isotropically into the upper hemisphere, we have to expect an isotropic contribution of this amount. This would mean a reduction of the isotropic part of the measured distribution from 20% to 6%. This correction was not applied, however, as the estimation made above is somewhat uncertain and its influence may be neglected.

The distribution can be approximated by

$$(9) \quad P(\vartheta) = 0.935 \cos^2 \vartheta + 0.857 \cos^3 \vartheta (1 - \cos \vartheta) + 0.023.$$

It is normalized according to

$$(10) \quad \int_{\vartheta=0}^{\pi} \int_{\varphi=0}^{2\pi} P(\vartheta) \sin \vartheta d\vartheta d\varphi = 1.$$

Angular distributions are as a rule described by a function of the form $\cos^n \vartheta$. Our experimental distribution cannot very well be described in this way; the best approximation is

$$(11) \quad P(\vartheta) = 0.53 \cos^6 \vartheta + 0.015.$$

Starting from the law of absorption of the primary protons we have

$$(12) \quad P(\vartheta) = P_0 \exp[-ax/\cos \vartheta],$$

$1/a$: absorption length in g/cm².

Our angular distribution can be described with satisfactory accuracy by equ. (12) with $ax=8.3$. On the altitude of the Zugspitze we have $x=700$ g/cm² and therefore $a^{-1}=85$ g/cm².

A better approximation would be

$$(13) \quad P(\vartheta) = 0.37 \cdot 10^4 \exp[-8.3/\cos \vartheta] - 0.05 \cos 4\vartheta + 0.08.$$

The term added to the form of equ. (12) is noticeable only for $\vartheta > 40^\circ$ (see Fig. 3).

⁽¹⁰⁾ E. CLEMENTEL and G. PUPPI: *Nuovo Cimento*, **8**, 936 (1951).

⁽¹¹⁾ E. CLEMENTEL and F. FERRARI: *Nuovo Cimento*, **9**, 572 (1952).

To examine the influence of the absorbers on the angular distribution, we compared the angular distribution of tracks from plates exposed at absorber depths < 5 cm with the total angular distribution (Fig. 4). There is no appreciable difference between the two distributions. We can conclude, that the angular distribution in air is not markedly different from the one observed in our absorbers. This was to be expected, since the form of our absorbers causes all particles reaching a certain plate to traverse approximately the same distance in the absorbers regardless of their angles of incidence. The error introduced by particles produced in the absorbers is small, too, since the absorbers are relatively small ($30 \div 40$ cm) and the shower particles are emitted approximately into the direction of the primary particle.

To investigate the influence of particle energy on the angular distribution we separated the particles into two groups according to the size of stars they produced. The high energy group

contained all particles producing a star with at least one shower particle or more than 7 grey or black tracks, the energy of these stars being > 1 GeV according to

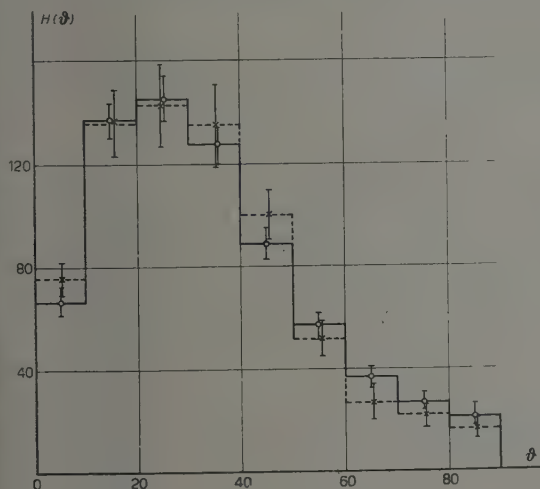


Fig. 4. — Influence of the absorbers on the angular distribution of star producing particles.

—: particles from all plates;

- - -: particles from plates exposed at a depth < 5 cm in the absorbers;

$H(\theta)$: arbitrary units.

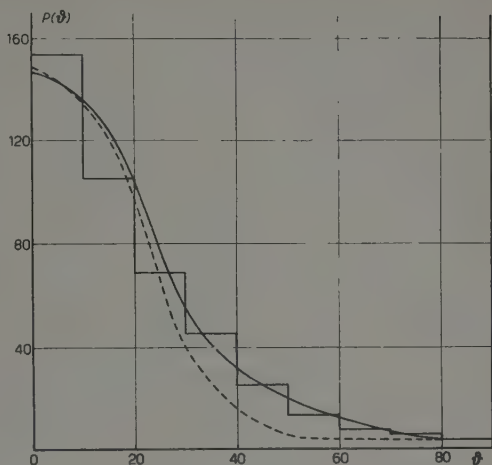


Fig. 3. — \neg angular distribution of ionizing star producing particles (at 2960 m); — approximation eq. (13) with $ar = 8.3$; - - - approximation eq. (12).

(¹²) R. H. BROWN, U. CAMERINI, P. H. FOWLER, H. HEITLER, D. T. KING and C. F. POWELL: *Phil. Mag.* **40**, 862 (1949).

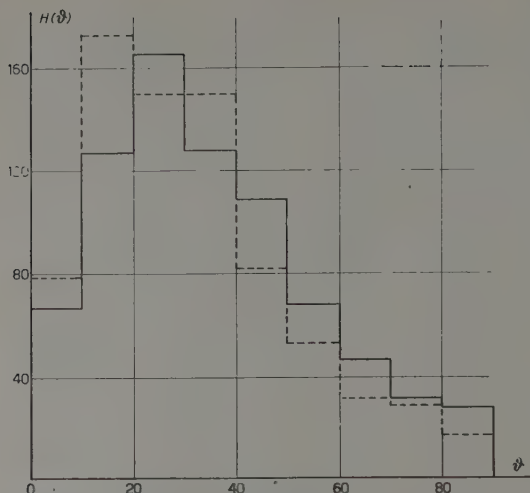


Fig. 5. - Influence of particle energy on the angular distribution of star producing particles.

: low energy group $E < 1$ GeV; high energy group $E > 1$ GeV; $H(\vartheta)$: arbitrary units.

al. ⁽¹²⁾). They found a distribution of the form

$$(14) \quad P(\vartheta) = P_0 \cos^{2.5} \vartheta.$$

This distribution does not well agree with our results, regarding the very good statistical accuracy of the distribution given by CONVERSI and ROTHWELL.

The measurements made by BARFORD and DAVIS ⁽¹⁴⁾, who found a $\cos^5 \vartheta$ distribution at an altitude of 3570 m agree with our own results.

4. - Distribution of Polar Angles of Ionizing Star Producing Particles at 420 m.

The angular distribution of ionizing star producing particles having an energy $E > 350$ MeV at 420 m is shown in Fig. 6. It can approximately be described by

$$(15) \quad P(\vartheta) = 0.84 \cos^{5.7} \vartheta + 0.03.$$

⁽¹³⁾ M. CONVERSI and P. ROTHWELL: *Nuovo Cimento*, **12**, 191 (1954) with extended list of further literature on angular distributions.

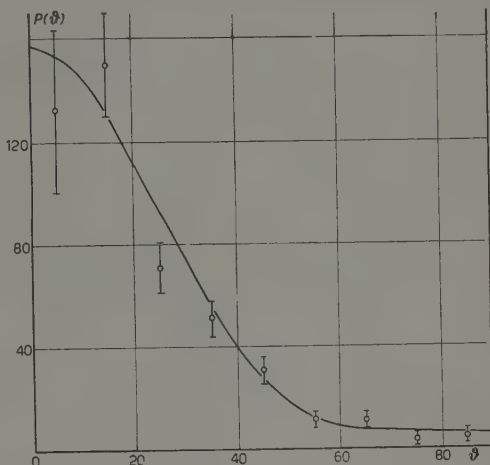
⁽¹⁴⁾ N. C. BARFORD and C. DAVIS: *Proc. Roy. Soc., A* **214**, 225 (1952).

BROWN *et al.* ⁽¹²⁾. The low energy group contained all particles having an energy E $350 \text{ MeV} < E < 1000 \text{ MeV}$. The angular distributions of the two groups are compared in Fig. 5. The high energy group seems to be more collimated in the direction of the vertical.

CONVERSI and ROTHWELL ⁽¹³⁾ have used a ionization chamber surrounded by G.M. counters to determine the angular distribution of ionizing star producing particles at an altitude of 3500 m. As a lower limit of their particle energy they give $E = 400$ MeV (determined from star size according to BROWN *et*

This is not in contradiction with measurements of COATES and HERZ⁽¹⁵⁾, who found a $\cos^2 \vartheta$ -distribution for «straight line tracks». The great majority of these tracks should be due to μ -mesons; this result is to be compared with measurements by LOVATI *et al.*⁽¹⁶⁾, who found the same $\cos^2 \vartheta$ -distribution by means of a cloud chamber at 3500 m; 88% of their particles were μ -mesons.

Fig. 6. — \bigcirc angular distribution of ionizing star producing particles having an energy $E > 350$ MeV at 420 m; —; approximation eq. (15); $P(\vartheta)$: arbitrary units.



5. — Angular Distribution of Grey Tracks relative to the Vertical.

The angular distribution of «grey» particles relative to the vertical having an energy E of $140 \text{ MeV} < E < 350 \text{ MeV}$ and $70 \text{ MeV} < E < 140 \text{ MeV}$ respectively at an altitude of 2960 m is shown in Fig. 7. This distribution agrees with the one used by RÖSSLE^(5,7) in his calculations on transition effects of the nucleonic component.

These «grey» particles can also produce stars. On substituting $180^\circ - \vartheta$ for ϑ one gets the angular distribution of star producing particles with an energy $E < 350$ MeV. This distribution is much less collimated

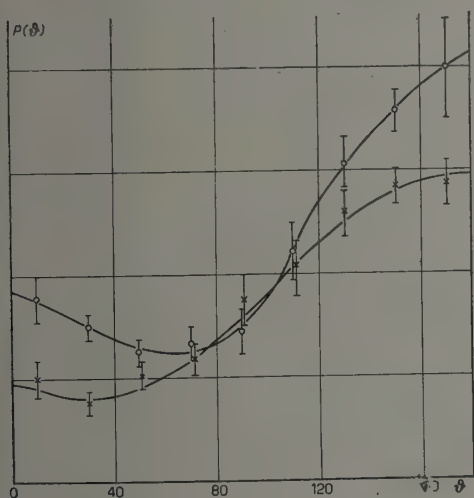


Fig. 7. — Angular distribution of grey tracks relative to the vertical at 2960 m; \bigcirc particle energy 140-350 MeV; \times particle energy 70-140 MeV.

⁽¹⁵⁾ A. C. COATES and R. H. HERZ: *Phil. Mag.*, **40**, 1088 (1949).

⁽¹⁶⁾ A. LOVATI, A. MURA, C. SUCCI and G. TAGLIAFERRI: *Nuovo Cimento*, **12**,

in the direction of the vertical and similar to the one found by CONVERSI and ROTHWELL for ionizing star producing particles. The disagreement between their result and our own equ. (9) could be understood, if these authors had overestimated the lower limit of their particle energy.

The relatively great fraction of particles apparently emitted backwards in Fig. 7 may in part be due to star producing grey particles which are mistaken for secondaries. This is also supported by Fig. 8, showing the angular distribution of grey tracks from stars formed by an ionizing particle with energy > 350 MeV. All these tracks must in this case be secondaries; accordingly the rise of $P(\vartheta)$ for $\vartheta < 90^\circ$ is missing.

We can exclude stars produced by grey particles by regarding only stars of an energy greater than the energy of the grey particle emerging from them (estimation of star energy according to BROWN *et al.* (12)).

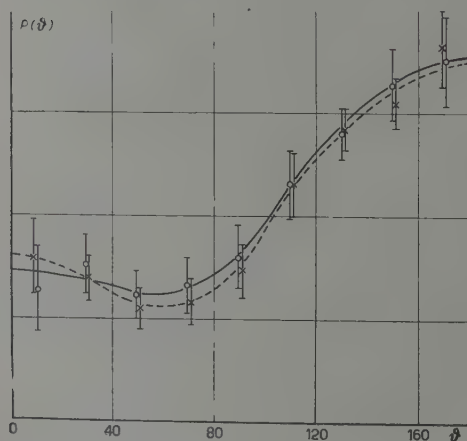


Fig. 9. — Angular distribution of grey particles from neutron induced stars. \times high energy stars; \times corrected distribution using all stars.

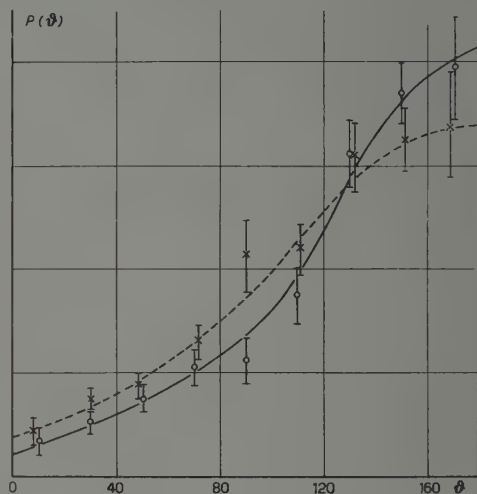


Fig. 8. — Angular distribution of grey tracks from stars produced by an ionizing particle at 2960 m. \circ energy of grey particle 140-350 MeV; \times energy of grey particle 70-140 MeV.

The full line in Fig. 9 shows the angular distribution of grey tracks in the energy interval $350 \text{ MeV} > E > 140 \text{ MeV}$ from these stars without a fast ionizing particle from the upper hemisphere. We can also estimate the fraction of star producing grey particles by regarding stars of low energy only. By subtracting this fraction from the total angular distribution of all stars without a visible primary track one gets the dotted line in Fig. 9. The two distributions are identical within the statistical limits. They represent the angular distribution of grey secondary particles emitted from stars produced by fast neutrons.

These measurements may be best compared with the results of CONVERSI and ROTHWELL⁽¹³⁾ as these authors use approximately the same energy intervals. There exists a satisfying agreement for the angular distribution of grey tracks emitted from stars produced by an ionizing particle. CONVERSI and ROTHWELL find the same distribution also for tracks from neutron-induced stars; this however is not in agreement with our own results.

6. - Angular Distribution of Grey Tracks relative to the Star Producing Particle.

Let ε be the angle between the star producing and the secondary particle; $\varepsilon = 180^\circ$ meaning a secondary running on in the same direction as the primary track.

Our definition of a star producing particle, meaning a particle having a track with grain density $g < 1.5g_{\min}$ and running in the upper hemisphere, can give rise to errors. The main error may arise from the same facts discussed in sec. 3 namely from stars having a shower particle in the upper hemisphere which can be mistaken for a star producing particle. The fraction of such stars was estimated to be 14% in sec. 3. As the grey particles are emitted approximately in the direction of the primary track, we have to expect a spurious part of tracks apparently emitted backwards. This can be seen by comparing our distribution with the one got by BERNARDINI⁽¹⁷⁾ *et al.* on artificially produced stars (Fig. 12).

The correction for losses of tracks due to dip (sec. 2) was carried out, but it is hardly important in this case.

The distribution of grey particles of our two energy intervals around the direction of the primary particle is shown in Fig. 10. With increasing energy of the secondaries they are emitted at smaller angles to the direction of the primary particle; this agrees with the results of CAMERINI *et al.*⁽¹⁸⁾.

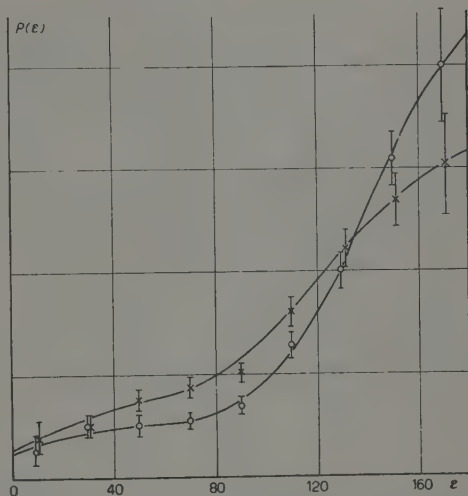


Fig. 10. - Angular distribution of grey tracks relative to the star producing particle. O energy of grey track 140-350 MeV; X energy of grey track 70-140 MeV.

⁽¹⁷⁾ G. BERNARDINI, E. T. BOOTH and S. J. LINDENBAUM: *Phys. Rev.*, **88**, 1017 (1952).

⁽¹⁸⁾ U. CAMERINI, J. H. DAVIES, P. H. FOWLER, C. FRANZINETTI, M. MUIRHEAD, W. O. LOCK, D. H. PERKINS and G. YEKUTIELI: *Phil. Mag.*, **42**, 1241 (1951).

The comparison of our measurements with the results of CAMERINI *et al.* ⁽¹⁸⁾ is shown in Fig. 11. Both distributions agree well, though the measurements of CAMERINI were made on stars from an altitude of 21 000 m. Noting that the average energy of star producing particles at 21 000 m is $2 \div 3$ times higher than at 3 000 m ^(9,18), we can conclude that the angular distribution of grey protons around the direction of the primary track is independent of its energy.

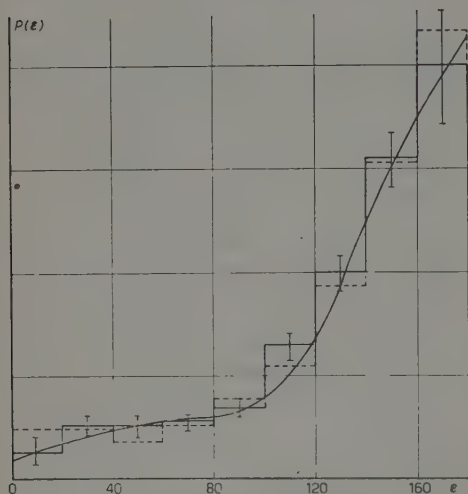


Fig. 11. — Angular distribution of grey tracks relative to the star producing particle. \square 140-350 MeV, own measurements (2 960 m); \square 130-280 MeV, CAMERINI ⁽¹⁸⁾ (21 000 m), the same statistical accuracy; — approximation eq. (18).

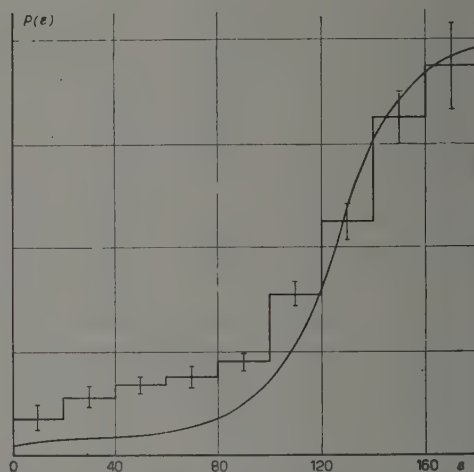


Fig. 12. — Angular distribution of grey tracks relative to the star producing particle. \square 70-350 MeV, own measurements; — 40-350 MeV, cyclotron stars observed by BERNARDINI *et al.*

This conclusion can be confirmed by comparing our results with the distribution of grey particles around the direction of star producing 350 \div 400 MeV protons from a cyclotron (BERNARDINI *et al.* ⁽¹⁷⁾) (Fig. 12). The shown spatial distribution was taken from a transformation carried out by CONVERSI and ROTHWELL, whereas BERNARDINI *et al.* they give the distribution of the projection of the angles on a plane lying in the primary beam. The distribution agrees with our own for $\epsilon > 90^\circ$, although the average primary energy is much lower. For $\epsilon < 90^\circ$ there is an excess of particles emitted backwards in our measurements, which amounts to 15%, corresponding to the estimation of systematic errors given above.

7. - Angular Distribution of Star Producing Neutrons.

The angular distribution of star producing neutrons cannot be found directly. We shall try to derive it from the angular distribution of their grey recoil protons.

Let $Q(\vartheta_2)d\Omega$ be the number of grey tracks and $P(\vartheta_1)d\Omega$ the number of star producing particles running into an element of solid angle $d\Omega$ and enclosing an angle of ϑ with the vertical. $R(\varepsilon)d\Omega$ denotes the distribution of grey tracks around the direction of the primary particle. We have then

$$(16) \quad Q(\vartheta_2) = \int_{\vartheta_1=0}^{\pi} \int_{\Phi=0}^{2\pi} P(\vartheta_1) R(\varepsilon) \sin \vartheta_1 d\vartheta_1 d\Phi,$$

noting that

$$(17) \quad \cos \varepsilon = \cos \vartheta_1 \cos \vartheta_2 + \sin \vartheta_1 \sin \vartheta_2 \cos \Phi, \quad \Phi = \varphi_1 - \varphi_2,$$

ϑ_1, φ_1 and ϑ_2, φ_2 are the polar and azimuthal angles of secondary and primary tracks respectively in the laboratory system.

To control the correction for dip equ. (6) and the angular distributions given so far we calculated the angular distribution of grey particles in the energy interval of $350 \div 140$ MeV relative to the vertical. Setting

$$(18) \quad R(\varepsilon) = 73 - 56 \cos \varepsilon + 158 \cos^2 \varepsilon - 156 \cos^3 \varepsilon$$

(see Fig. 11) and choosing for $P(\vartheta_1)$ the equ. (9) we get for $Q(\vartheta_2)$ from equ. (16)

$$(19) \quad Q(\vartheta_2) = 499 - 446 \cos \vartheta_2 + 385 \cos^2 \vartheta_2 - 213 \cos^3 \vartheta_2.$$

The integral was evaluated by substituting

$$(20) \quad \cos \vartheta_1 = x \quad \text{and} \quad \cos \vartheta_2 = y$$

leading to

$$(21) \quad \bar{Q}(y) = \int_{x=-1}^{+1} \bar{P}(x) \bar{R}(x, y) dx;$$

$\bar{P}(x)$ is a polynomial of x and $\bar{R}(x, y)$ an algebraic function of x and y .

The calculated distribution equ. (19) agrees well with the distribution measured directly as is shown in Fig. 13.

We can now try to calculate the angular distribution $P(\vartheta_1)$ of the primary

particles, from the known functions $R(\varepsilon)$ and $Q(\vartheta_2)$. By means of the substitution equ. (20) the problem is reduced to a Fredholm integral equation of the first kind for $\bar{P}(x)$. It can easily be shown, that this equation does not give a unique solution for $\bar{P}(x)$ in practically important cases. Let $\bar{P}_1(x)$ and $\bar{P}_2(x)$ be two distributions fulfilling equ. (21); we have for their difference $\Delta\bar{P}(x) = P_1(x) - P_2(x)$:

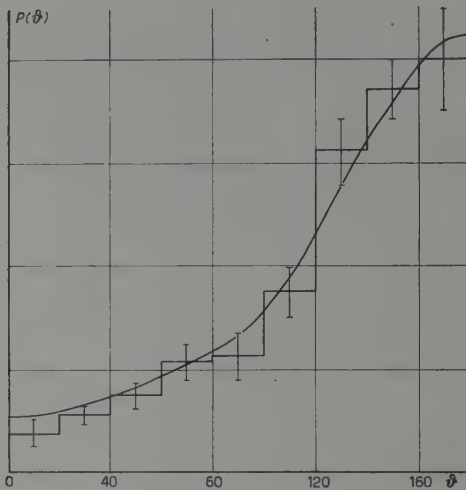


Fig. 13.—Angular distribution of grey particles having an energy of 140-350 MeV from stars induced by an ionizing particle relative to the vertical. \updownarrow measured (compare fig. 8); — calculated according to eq. (19).

solution of the integral equation (16), another one would be

$$(23) \quad P_2(\vartheta_1) = P_1(\vartheta_1) + \lambda \Delta P(\vartheta_1),$$

λ being an arbitrary parameter. As $R(x, y)$ is always > 0 , P must have at least one zero as follows from equ. (22). Therefore λ has an upper limit, as $P_2(\vartheta_1)$ must always be > 0 too. Assuming the function $P(\vartheta_1)$ to be of the form $\cos^n \vartheta_1$, we can at least derive a statement about the exponent n by comparing the function $Q(\vartheta_1)$ measured with the functions calculated for different values of n . By means of this method we estimated the angular distribution of star producing neutrons having an energy $E > 350$ MeV. We assumed the angular distribution of grey tracks around the direction of the star producing particle to be equal for neutron and proton induced stars, so we used equ. (18) for $R(\varepsilon)$ also in this case.

We calculated $Q(\vartheta_2)$ for two trial neutron distributions:

$$(24) \quad P(\vartheta_1) = \begin{cases} \cos^4 \vartheta_1 + 0.043 & \text{for } 0 < \vartheta_1 < 90^\circ \\ 0.043 & \text{for } 90 < \vartheta_1 < 180^\circ \end{cases}$$

$$(22) \quad 0 \equiv \int_{x=-1}^{+1} \Delta\bar{P}(x) \bar{R}(x, y) dx.$$

The integral equ. (22) leads to a polynomial in y . If this polynomial is of n -th degree, we need in general n adjustable parameters in the function $\Delta\bar{P}(x)$ to cause the integral to vanish identically, that means that equ. (22) has a not trivial solution.

It can be shown however that there exists an upper limit for the function $\Delta P(\vartheta_1)$. If $P_1(\vartheta_1)$ is a

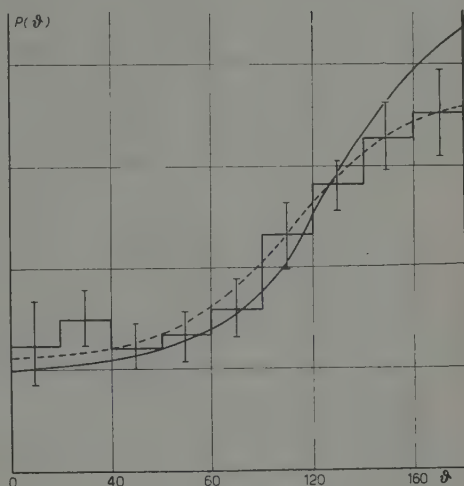
$$(25) \quad P(\vartheta_1) = \begin{cases} \cos \vartheta_1 + 0.11 & \text{for } 0 < \vartheta_1 < 90^\circ \\ 0.11 & \text{for } 90^\circ < \vartheta_1 < 180^\circ. \end{cases}$$

The fraction of neutrons coming from the lower hemisphere is 15%. The distributions $Q(\vartheta_2)$ calculated by means of eq. (16) are to be compared with the one found for grey tracks coming from high energy stars (cf. Fig. 9). As is shown in Fig. 14, the neutron angular distribution should not be more collimated than is indicated by eq. (24).

The best fit is obtained by eq. (25).

This result agrees with measurements made by BARFORD and DAVIS⁽¹⁴⁾ and also by CONVERSI and ROTHWELL⁽¹³⁾ within the large statistical errors.

Fig. 14. - Angular distribution of grey tracks (140÷350 MeV) from neutron induced stars. \square measured (compare fig. 9); — calculated using the angular distribution for neutrons according to eq. (24); --- calculated using angular distribution eq. (25).



Acknowledgements.

The author wishes to express his thanks to Prof. Dr. E. SCHOPPER and to Dr. E. RÖSSELE for continuous encouragement and valuable discussions. The financial help of the Deutsche Forschungsgemeinschaft is gratefully acknowledged.

RIASSUNTO (*)

La distribuzione angolare delle particelle generatrici di stelle e dei loro secondari è stata misurata sotto assorbitori di Fe, Sn e C a 2960 m (Zugspitze) e 420 m (Weis-

(*) Traduzione a cura della Redazione.

senau). La distribuzione angolare delle particelle ionizzanti generatrici di stelle può essere rappresentata approssimativamente da una legge $\cos^6 \vartheta$ a 2960 m e $\cos^{5.7} \vartheta$ a 420 m. Al crescere dell'energia delle particelle la distribuzione appare più fortemente collimata verso la verticale. La distribuzione angolare delle tracce grigie intorno alla direzione delle particelle generatrici di stelle si accorda coi risultati del gruppo di Bristol per altezze stratosferiche, facendo ritenere che la distribuzione sia indipendente dall'energia della particella generatrice di stelle. Al crescere dell'energia si nota un'accresciuta collimazione della distribuzione angolare. La distribuzione angolare delle tracce grigie rispetto alla verticale è data per le stelle prodotte da neutroni e, rispettivamente da protoni. Si è fatto un tentativo di derivare la distribuzione angolare della componente neutra generatrice di stelle giungendo a una legge $\cos^n \vartheta$ con $1 < n < 4$.

On the Topographic Distribution of Cosmic Ray Stars in Nuclear Emulsion.

E. LOHRMANN

*Hochspannungslaboratorium Hechingen
des Max Planck-Instituts für Physik der Stratosphäre*

(ricevuto il 28 Marzo 1955)

Summary. — Cosmic ray stars are found to have a completely random distribution at distances $< 500 \mu$ in nuclear research emulsions, which had been exposed under different absorbers at mountain altitudes. This random distribution is not influenced by transition effects of the star frequency.

In the course of an investigation by SCHOPPER *et al.* ^(1,2) on the absorption of the star producing components of the cosmic radiation we have analysed the distribution of stars in nuclear research emulsions. These plates had been exposed under solid absorbers of Fe, Sn and C for $40 \div 50$ days at 2960 m altitude (Zugspitze); their examination showed transition effects giving rise to maximum values of star frequency as a function of depth in the absorbers.

A paper of LEPRINCE-RINGUET and HEIDMANN ⁽³⁾ reporting an abnormal frequency of cosmic ray stars with a distance $< 500 \mu$ suggested some sort of irregularity of the star producing component. Furthermore LI ⁽⁴⁾ had reported a similar effect in plates exposed at mountain altitudes under absorbers. So we were induced to investigate a possible relation between our above men-

⁽¹⁾ E. SCHOPPER, K. H. HÖCKER and E. RÖSSLE: *Zeits. f. Naturf.*, **6a**, 603 (1951).

⁽²⁾ E. RÖSSLE and E. SCHOPPER: *Zeits. f. Naturf.*, **9a**, 836 (1954).

⁽³⁾ L. LEPRINCE-RINGUET and J. HEIDMANN: *Nature*, **161**, 844 (London, 1948).

⁽⁴⁾ T. T. LI: *Phil. Mag.*, **41**, 1152 (1950).

tioned maxima of star frequency and the anomalous distribution of stars, for there are similar difficulties to explain both effects.

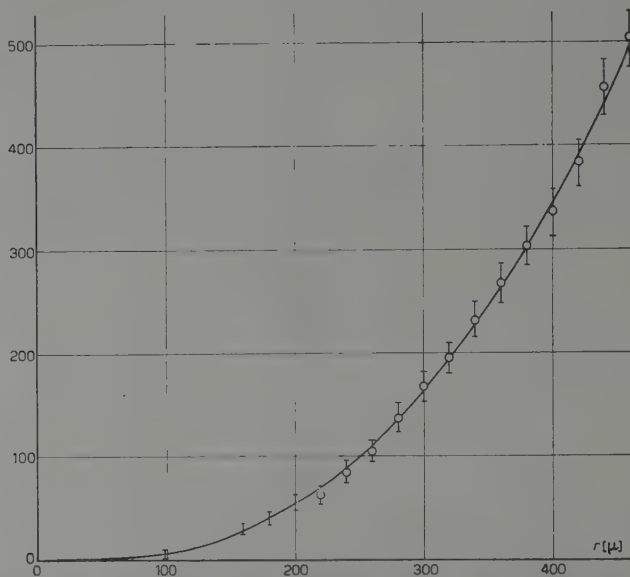


Fig. 1. — Distribution of stars in the emulsions of all plates, \bar{Q} observed number of pairs of stars $P(r)$ (corrected) — expectation value of number of pairs of stars $Q(r)$.

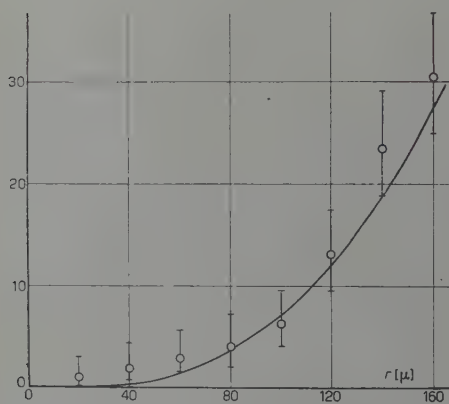


Fig. 1a. — Distribution of stars for small distances.

The large number of Ilford G 5 nuclear research plates, size 5×5 cm, emulsion thickness 500μ , exposed by SCHOPPER *et al.*, was available for this purpose. The plates were scanned by rows; all « double stars » found in this way were noted, « double star » meaning two stars being visible in the same visual field of the microscope and not connected by a track. An area of 472 cm^2 containing 9364 stars was examined.

Let $P(r)$ be the number of « double stars » being in this area having a distance $\leq r$, and r_p the distance of the stars projected into the plane of the plate. On account of our scanning by rows we can expect to find only the fraction

$$(1) \quad p = \frac{2}{\pi} \arccos \left(\frac{r_p}{2R} \right)$$

on these stars, if $2R$ means the diameter of field of view.

This loss was accounted for in all cases, the correction being small for distances $< 300 \mu$.

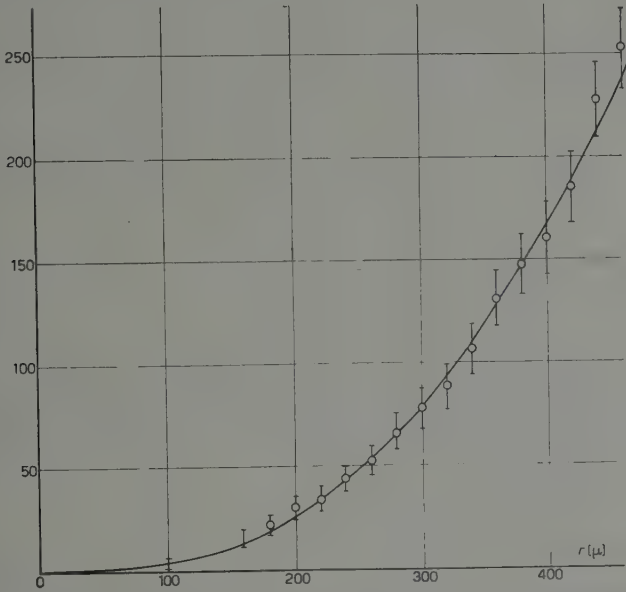


Fig. 2. — Distribution of stars in plates exposed at points of maximum star frequency. \bigcirc observed number of pairs $P(r)$; — expectation value of number of pairs $Q(r)$.

Let $Q(r)$ be the expectation value of the number of « double stars » having a distance $\leq r$. Assuming the stars to be randomly distributed, $Q(r)$ can be shown to be (see appendix).

$$(2) \quad Q(r) = \sum_i \frac{N_i(N_i - 1)}{2F_i} \cdot \begin{cases} \frac{4\pi}{3d} r^3 \left(1 - \frac{3}{8} \frac{r}{d}\right) & \text{for } r \leq d \\ \pi r^2 \left[1 - \frac{1}{6} \left(\frac{d}{r}\right)^2\right] & \text{for } r \geq d \end{cases}$$

N_i : number of stars in plate number i having an area of F_i
 d : emulsion thickness. The sum is extended over all plates.

A comparison between $Q(r)$ and the actual number $P(r)$ of « double stars » is shown in Fig. 1 and 1a. The two distributions are identical within the limits of the statistical errors.

Confining this analysis to photographic plates which had been exposed at

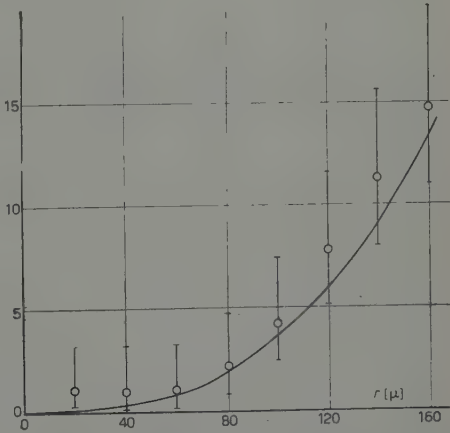


Fig. 2a. — Distribution of stars for small distances at points of maximum star frequency.

points of maximum star frequency in the absorbers, we found also an agreement between the number of « double stars » $P(r)$ found in the plates and their expectation value $Q(r)$, as is shown in Fig. 2 and 2a. We can therefore conclude that (i) the stars are randomly distributed in the emulsion at distances $< 500 \mu$ and (ii) that this distribution is not influenced by transition effects.

APPENDIX

Derivation of equ. (2).

In an infinitely extended emulsion the number of stars having a distance $\leq r$ from a given star would be

$$(3) \quad n(r) = \frac{4\pi}{3} r^3 \cdot \varrho_i ;$$

ϱ_i is the « star density »

$$(4) \quad \varrho_i = \frac{N_i - 1}{V_i} ,$$

N_i means the number of stars in a volume V_i .

Summing over all plates having an area F and a thickness of d and over all stars one finds the number of pairs of stars having a distance $\leq r$

$$(5) \quad \bar{Q}(r) = \sum_i \frac{(N_i - 1)N_i}{2 \cdot F_i \cdot d} \cdot 4\pi r^3 .$$

To calculate the influence of finite emulsion thickness d consider a star having a distance x from the surface of the emulsion. The number of stars having a distance $\leq r$ from this star is proportional to that part $V(x, r, d)$ of the volume of a sphere around this star with radius r , which lies in the emulsion. Taking the mean over the values of x , one has

$$(6) \quad \bar{V}(r, d) = \frac{1}{d} \int_{x=0}^d V(x, r, d) dx ,$$

$\bar{V}(r, d)$ is the mean value of the volume where we have stars with a distance $\leq r$ from a given star. Substituting \bar{V} for $4\pi r^3/3$ in equ. (5) one gets equ. (2).

Acknowledgements.

I would like to express my gratitude to Prof. Dr. E. SCHOPPER and to Dr. E. RÖSSLE for continued encouragement and stimulating discussions. The author was supported by a grant from the Deutsche Forschungsgemeinschaft.

RIASSUNTO (*)

Si trova che le stelle prodotte da raggi cosmici in emulsioni nucleari esposte sotto differenti assorbitori ad altitudini di montagna hanno una distribuzione completamente casuale a distanze $< 500 \mu$. Tale distribuzione casuale non è influenzata da effetti di transizione della frequenza delle stelle.

(*) *Traduzione a cura della Redazione.*

Statistical Weights in Many-Particle Systems.

Y. YEIVIN and A. DE SHALIT

The Weizmann Institute of Science - Rehovoth, Israel

(ricevuto il 29 Marzo 1955)

Summary. — A general formula for the statistical weights $G_m(S)$ of the states, with total spin S , of a system of r spin $\frac{1}{2}$ and n spin 1 particles is derived by combinatorial methods.

1. — Introduction.

In dealing with many-particle systems we often do not require all the parameters (quantum numbers) which completely characterize a state of the system. If the property considered depends only on some of these parameters the others do not contribute at all to the solution of the problem, except to enumerate the state vectors. In treating problems of this sort it may, therefore, be useful to develop a method for evaluating the statistical weights (multiplicities) of states characterized by an incomplete set of parameters.

A common case encountered is that of a system of identical particles, where the only parameters of importance for their states are the total spin (either angular momentum or isotopic spin) and its projection on a certain axis. Thus, for instance, in considering meson production in nucleon-nucleon collisions one may assume that the isotopic spin of the system is conserved. However, if no further information about the interaction is available, one cannot determine a definite final state out of the totality of states of the two nucleons and produced mesons consistent with the assumption of isotopic-spin conservation. Consequently, equal probabilities are assumed for all these states, and the «state characterized by T and M_T » is given a statistical weight equaling the number of independent combinations by which it can be formed.

Usually, it is not difficult to calculate the statistical weights of such states, especially for a small number of particles, since use may be made of trivial

recurrence formulas. When, however, the number of particles becomes large, and, again, in treatments which claim a more general validity, it is sometimes more convenient to have at hand closed formulas for these weights.

The purpose of this note is to give explicit solutions for the recurrence formulas, and thus derive a general formula for the statistical weights of the states of any number of spin $-\frac{1}{2}$ and spin -1 particles with a definite total spin. A similar formula for some special cases has been obtained by RACAH ⁽¹⁾ using group-theoretical methods.

2. - Homogeneous Systems.

2.1. Spin $\frac{1}{2}$ particles. - We shall first confine ourselves to systems consisting of just one type of particle, starting with particles of spin $\frac{1}{2}$. Since the number of independent states is independent of the scheme, we may choose that particular scheme in which the z -component of the spin of each particle separately is diagonal. For spin $\frac{1}{2}$ particles the z -component of each spin can be either $+\frac{1}{2}$ or $-\frac{1}{2}$, and we immediately see that in any state, with $S_z = M$, of such particles $\frac{1}{2}\nu + M$ will have their spins up, while $\frac{1}{2}\nu - M$ will have theirs down. As there are $\binom{\nu}{\frac{1}{2}\nu - M}$ ways of making this arrangement, it is clear that for spin $\frac{1}{2}$ particles this is the number of states of total spin-projection M . These states as they stand are not, however, eigenstates of S^2 . Instead, this group of $\binom{\nu}{\frac{1}{2}\nu - M}$ states contains (linear) combinations of all the states characterized by the fixed M and by any $S \geq M$. Therefore ⁽²⁾, it follows that the number of states with a given S and M (where $|M| \leq S$) is:

$$(1) \quad \gamma_\nu(SM) = \binom{\nu}{\frac{1}{2}\nu - S} - \binom{\nu}{\frac{1}{2}\nu - (S+1)} = \frac{2S+1}{\nu+1} \binom{\nu+1}{\frac{1}{2}\nu - S}.$$

It is clear from (1) that $\gamma_\nu(SM)$ is independent of M , as should be anticipated. We shall therefore omit the M .

One may readily convince oneself that (1) is a solution of the recurrence formulas which $\gamma_\nu(S)$ should obviously satisfy, namely:

$$(2) \quad \begin{cases} \gamma_{\nu+1}(S) = \gamma_\nu(S - \frac{1}{2}) + \gamma_\nu(S + \frac{1}{2}) \dots S \neq 0 \\ \gamma_{\nu+1}(0) = \gamma_\nu(\frac{1}{2}). \end{cases}$$

⁽¹⁾ G. RACAH: *Rend. Lincei*, **17**, 386 (1933).

⁽²⁾ See, for example, E. U. CONDON and G. H. SHORTLEY: *The Theory of Atomic Spectra* (London, 1952), second edition, p. 189.

2.2. *Spin 1 particles.* — In the case of a homogeneous system of spin 1 particles the recurrence formulas for the number of states of total spin S and any fixed value of M are

$$(3) \quad \begin{cases} g_{n+1}(S) = g_n(S-1) + g_n(S) + g_n(S+1) \dots S \neq 0 \\ g_{n+1}(0) = g_n(1) . \end{cases}$$

These equations can be most easily solved if one notes that a system of n spin 1 particles may be considered, in some respects, as a system of $\nu = 2n$ spin $\frac{1}{2}$ particles. Let us define $\gamma_{2n}(S; i)$ as the number of states of $2n$ spin $\frac{1}{2}$ particles with a total S (and M) in which i pairs are each coupled to spin 1 and $n-i$ to spin 0 ⁽³⁾. Since, now, there are $\binom{n}{i}$ ways of choosing the i pairs, we see that

$$(4) \quad \gamma_{2n}(S; i) = \binom{n}{i} g_i(S) ,$$

where $g_i(S)$ is defined in (3). Therefore

$$(5) \quad \gamma_{2n}(S) = \sum_i \binom{n}{i} g_i(S) .$$

This equation can be solved for $g(S)$ in terms of $\gamma(S)$ by using the following orthogonality relation of the binomial coefficients:

$$(6) \quad \sum_k (-)^{j+k} \binom{j}{k} \binom{k}{i} = \delta_{ij} .$$

(This relation is most easily proved by noting that the identity

$$x^j = [1 - (1-x)]^j = \sum_{ik} (-)^{i+k} \binom{j}{k} \binom{k}{i} x^i$$

holds for every value of x .) Substituting for $\gamma(S)$ from (1) we then get

$$(7) \quad g_n(S) = (2S+1) \sum_i \frac{(-)^{n+i}}{2i+1} \binom{n}{i} \binom{2i+1}{i-S} .$$

⁽³⁾ The previously defined weights $\gamma_{2n}(S)$ satisfy $\gamma_{2n}(S) = \sum_i \gamma_{2n}(S; i)$.

3. — Mixed Systems.

Any system of particles with a total spin S may be always decomposed into two partial systems with spins S' and S'' , where $S' + S'' = S$. To calculate the « compound » weight in terms of the « partial » weights one has to consider all possible values of S' and S'' satisfying the triangular condition with S . Therefore, the weight $G_{\nu n}(S)$ of a state of a system of ν spin $\frac{1}{2}$ particles and n spin 1 particles is given in terms of the weights $\gamma_{\nu}(S')$ and $g_n(S'')$ of the preceding section by

$$(8) \quad G_{\nu n}(S) = \sum_{S' S''} \gamma_{\nu}(S') g_n(S'').$$

If ν is even, we can use equation (5) to obtain

$$G_{2k, n}(S) = \sum_{S' S''} \binom{k}{i} g_i(S') g_n(S'').$$

However, $\sum_{S' S''} g_i(S') g_n(S'')$ may also be interpreted as the statistical weight of a state S of $i+n$ spin 1 particles, since this weight could have been obtained by splitting the group of $i+n$ particles into two groups, of i and n particles respectively, and summing over all possible combinations. Hence

$$G_{2k, n}(S) = \sum_i \binom{k}{i} g_{i+n}(S),$$

or, by equation (7),

$$G_{2k, n}(S) = (2S + 1) \sum_{ij} \frac{(-)^{n+i+j}}{2j+1} \binom{k}{i} \binom{i+n}{j} \binom{2j+1}{j-S}.$$

Using the orthogonality relation (6) and other well known identities, and recalling that for our special case $k = \frac{1}{2}\nu$ (ν even), we obtain

$$(9) \quad G_{\nu n}(S) = (2S + 1) \sum_i \frac{(-)^{n+i}}{2i + \nu + 1} \binom{n}{i} \binom{2i + \nu + 1}{i + \frac{1}{2}\nu - S}.$$

We have proved this formula for even ν 's only. That it is correct for odd ν 's as well can be verified with the help of the first of equations (2). Thus (9) is our final general formula which gives an explicit expression for the weight

TABLE I. — *Some weights $G_m(S)$ of states with total spin S of ν spin $-\frac{1}{2}$ and n spin -1 particles, as given by equation (9).*

| S ν n | 0 | | | 1/2 | | | 1 | | | 3/2 | | |
|---------------------|-----|------|------|------|------|-------|------|------|-------|------|-------|-------|
| | 0 | 2 | 4 | 1 | 3 | 5 | 0 | 2 | 4 | 1 | 3 | 5 |
| 0 | 1 | 1 | 2 | 1 | 2 | 5 | — | 1 | 3 | — | 1 | 4 |
| 1 | 0 | 1 | 3 | 1 | 3 | 9 | 1 | 2 | 6 | 1 | 3 | 10 |
| 2 | 1 | 2 | 6 | 2 | 6 | 19 | 1 | 4 | 13 | 2 | 7 | 24 |
| 3 | 1 | 4 | 13 | 4 | 13 | 43 | 3 | 9 | 30 | 5 | 17 | 59 |
| 4 | 3 | 9 | 30 | 9 | 30 | 102 | 6 | 21 | 72 | 12 | 42 | 148 |
| 5 | 6 | 21 | 72 | 21 | 72 | 250 | 15 | 51 | 178 | 30 | 106 | 378 |
| 6 | 15 | 51 | 178 | 51 | 178 | 628 | 36 | 127 | 450 | 76 | 272 | 980 |
| 7 | 36 | 127 | 450 | 127 | 450 | 1608 | 91 | 323 | 1158 | 196 | 708 | 2573 |
| 8 | 91 | 323 | 1158 | 323 | 1158 | 4181 | 232 | 835 | 3023 | 512 | 1865 | 6828 |
| 9 | 232 | 835 | 3023 | 835 | 3023 | 11009 | 603 | 2188 | 7986 | 1353 | 4963 | 18286 |
| 10 | 603 | 2188 | 7986 | 2188 | 7986 | 29295 | 1585 | 5789 | 21309 | 3610 | 13323 | 49320 |

| S ν n | 2 | | | 5/2 | | | 3 | | | 7/2 | | |
|---------------------|------|------|-------|------|-------|-------|------|------|-------|------|-------|-------|
| | 0 | 2 | 4 | 1 | 3 | 5 | 0 | 2 | 4 | 1 | 3 | 5 |
| 0 | — | — | 1 | — | — | 1 | — | — | — | — | — | — |
| 1 | — | 1 | 4 | — | 1 | 5 | — | — | 1 | — | — | 1 |
| 2 | 1 | 2 | 11 | 1 | 4 | 16 | — | 1 | 5 | — | 1 | 6 |
| 3 | 2 | 8 | 29 | 3 | 12 | 46 | 1 | 4 | 17 | 1 | 5 | 23 |
| 4 | 6 | 21 | 76 | 9 | 34 | 128 | 3 | 13 | 52 | 4 | 18 | 76 |
| 5 | 15 | 55 | 200 | 25 | 94 | 352 | 10 | 39 | 152 | 14 | 58 | 235 |
| 6 | 40 | 145 | 530 | 69 | 258 | 965 | 29 | 113 | 435 | 44 | 177 | 702 |
| 7 | 105 | 385 | 1415 | 189 | 707 | 2647 | 84 | 322 | 1232 | 133 | 525 | 2057 |
| 8 | 280 | 1030 | 3805 | 518 | 1940 | 7277 | 238 | 910 | 3472 | 392 | 1532 | 5960 |
| 9 | 750 | 2775 | 10300 | 1422 | 5337 | 20065 | 672 | 2562 | 9765 | 1140 | 4428 | 17154 |
| 10 | 2025 | 7525 | 28051 | 3915 | 14728 | 55505 | 1890 | 7203 | 27454 | 3288 | 12726 | 49180 |

| S ν n | 4 | | | 9/2 | | | 5 | | | 11/2 | | |
|---------------------|------|------|-------|------|------|-------|-----|------|-------|------|------|-------|
| | 0 | 2 | 4 | 1 | 3 | 5 | 0 | 2 | 4 | 1 | 3 | 5 |
| 2 | — | — | 1 | — | — | 1 | — | — | — | — | — | — |
| 3 | — | 1 | 6 | — | 1 | 7 | — | — | 1 | — | — | 1 |
| 4 | 1 | 5 | 24 | 1 | 6 | 31 | — | 1 | 1 | — | 1 | 8 |
| 5 | 4 | 19 | 83 | 5 | 25 | 115 | 1 | 6 | 32 | 1 | 7 | 40 |
| 6 | 15 | 64 | 267 | 20 | 90 | 390 | 5 | 26 | 123 | 6 | 33 | 164 |
| 7 | 49 | 203 | 825 | 70 | 300 | 1256 | 21 | 97 | 431 | 27 | 131 | 604 |
| 8 | 154 | 622 | 2488 | 230 | 956 | 3917 | 76 | 334 | 1429 | 104 | 473 | 2084 |
| 9 | 468 | 1866 | 7389 | 726 | 2961 | 11961 | 258 | 1095 | 4572 | 369 | 1611 | 6890 |
| 10 | 1398 | 5523 | 21276 | 2235 | 9000 | 36005 | 837 | 3477 | 14279 | 1242 | 5279 | 22120 |

TABLE I. - (Continued).

| S | 6 | | | 13/2 | | | 7 | | | 15/2 | | |
|-----|-------|------|------|------|------|-------|-----|-----|------|------|------|------|
| | ν | | | | | | | | | | | |
| n | 0 | 2 | 4 | 1 | 3 | 5 | 0 | 2 | 4 | 1 | 3 | 5 |
| 4 | — | — | 1 | — | — | 1 | — | — | — | — | — | — |
| 5 | — | 1 | 8 | — | 1 | 9 | — | — | 1 | — | — | 1 |
| 6 | 1 | 7 | 41 | 1 | 8 | 50 | — | 1 | 9 | — | 1 | 10 |
| 7 | 6 | 34 | 173 | 7 | 42 | 224 | 1 | 8 | 51 | 1 | 9 | 61 |
| 8 | 28 | 139 | 655 | 35 | 182 | 889 | 7 | 43 | 234 | 8 | 52 | 296 |
| 9 | 111 | 516 | 2318 | 147 | 707 | 3269 | 36 | 191 | 951 | 44 | 244 | 1258 |
| 10 | 405 | 1802 | 7841 | 560 | 2562 | 11417 | 155 | 760 | 3576 | 200 | 1014 | 4908 |

| S | 8 | | | 17/2 | | | 9 | | | 19/2 | | |
|-----|-------|-----|------|------|-----|------|---|----|-----|------|----|-----|
| | ν | | | | | | | | | | | |
| n | 0 | 2 | 4 | 1 | 3 | 5 | 0 | 2 | 4 | 1 | 3 | 5 |
| 6 | — | — | 1 | — | — | 1 | — | — | — | — | — | — |
| 7 | — | 1 | 10 | — | 1 | 11 | — | — | 1 | — | — | 1 |
| 8 | 1 | 9 | 62 | 1 | 10 | 73 | — | 1 | 11 | — | 1 | 12 |
| 9 | 8 | 53 | 307 | 9 | 63 | 381 | 1 | 10 | 74 | 1 | 11 | 86 |
| 10 | 45 | 254 | 1332 | 54 | 318 | 1725 | 9 | 64 | 393 | 10 | 75 | 480 |

| S | 10 | | | 21/2 | | | 11 | | 23/2 | | 12 | 25/2 |
|-----|-------|----|----|------|----|-----|----|----|------|----|----|------|
| | ν | | | | | | | | | | | |
| n | 0 | 2 | 4 | 1 | 3 | 5 | 2 | 4 | 3 | 5 | 4 | 5 |
| 8 | — | — | 1 | — | — | 1 | — | — | — | — | — | — |
| 9 | — | 1 | 12 | — | 1 | 13 | — | 1 | — | 1 | — | — |
| 10 | 1 | 11 | 87 | 1 | 12 | 100 | 1 | 13 | 1 | 14 | 1 | 1 |

of a state (SM) in systems consisting of ν spin $\frac{1}{2}$ and n spin 1 particles. Table I gives some values of the $G_{\nu n}(S)$.

An application of equation (9) to calculations in multiple meson production will be given elsewhere.

RIASSUNTO (*)

Con metodo combinatorio si deriva una formula generale per i pesi statistici $G_{\nu n}(S)$ degli stati, con spin totale S , di un sistema di ν particelle con spin $\frac{1}{2}$ e n con spin 1.

(*) Traduzione a cura della Redazione

Physics and Mathematical Logic.

U. FARINELLI and A. GAMBA

Istituto di Fisica dell'Università - Torino

(ricevuto il 31 Marzo 1955)

Summary. -- The possibility of application of mathematical logic to the investigation of physical problems is discussed; the basic elements of mathematical logic are introduced and examples are given.

1. -- Introduction.

The puzzles one usually finds in cross-words magazines seem to be of two different types. Let us give an example for each type.

Problem 1. What is the number the square of which is equal to one unity less than the double of 25?

Problem 2. There are three tribes, A , B , C ; the men of tribe A always lie, those of tribe B always say the truth, whereas the men of tribe C alternately lie and say the truth (of course one cannot know whether the first answer is wrong or right). An explorer finds three men, a , b , c , who may belong to any of these tribes and asks to each man the following three questions in this order: 1) Which is the tribe of a ? 2) Which is the tribe of b ? 3) Which is the tribe of c ? The answers he receives are as follows:

TABLE I.

| replies of | to 1-st question (tribe of a) | to 2-nd question (tribe of b) | to 3-rd question (tribe of c) |
|------------|-------------------------------------|-------------------------------------|-------------------------------------|
| a | C | A | A |
| b | B | C | A |
| c | A | B | C |

To which tribe do a , b , c belong respectively?

The first is a trivial algebraic problem. The second one (at least according to the impressions one gets from lunch-time conversations of physicists) seems to be more difficult. However, the two kinds of problems would more or less show the same degree of difficulty if the physicists were acquainted with mathematical logic as much as they are with ordinary mathematics. The reason for this is that apparently most if not all problems in physics are of the first kind. But why should it be so?

In order to discuss this question, the formalism of mathematical logic will be briefly outlined in section 2; sections 3 and 4 will give an example of how a puzzle and a physical problem respectively can be dealt with by means of mathematical logic.

2. - The Formalism of Mathematical Logic ⁽¹⁾.

We shall indicate with a letter, A , B , C , ... any statement, as, for example, «the horse is white», «the triangle has four sides», etc.. We shall only consider statements which are either true or false. The formula $A=1$ will be given the meaning «statement A is true», whereas the formula $A=0$ means «statement A is false». For example, if B means «the sun is smaller than the moon» we write $B=0$.

Let us now define the product of two statements, AB , as meaning «statement A and statement B ». This is consistent with the usual rules of algebra, since $AB=1$ (both statements are true) only if $A=B=1$; and $AB=0$ (both statements are not true at the same time) in the following three cases: $A=0$, $B=1$; $A=1$, $B=0$; $A=B=0$. Therefore the logical «and» satisfies to the same rules as the ordinary product of algebraic quantities.

Similarly, the word «or» has the same properties as the $+$ sign of algebra, provided that we introduce the following limitation: $1+1=1$. In fact, if $A+B=1$ means «either statement A or statement B is true», this is so when both are true ($1+1=1$), when A is true and B is false ($1+0=1$), and when A is false and B is true ($0+1=1$). In the same way, if $A+B=0$ means that either by giving statement A or statement B one always says the false, this is so only when both A and B are false ($0+0=0$).

The algebra in which only numbers 0 and 1 exist and that satisfies the usual rules of arithmetics except $1+1=1$ is known as Boolean algebra.

A more general form can be given to the theory by introducing the concept of *classes*. Symbols like A , B , C , ... are now operators which select certain classes out of the universe of the objects we consider. For instance,

⁽¹⁾ See, for instance, G. BOOLE: *The mathematical analysis of logic* (Oxford, 1948) which is yet one of the clearest books among the vast literature.

A may be the operator which takes out of the universe the «white horses», B the «odd numbers», C the «neutral pions», etc..

The universe is indicated by the simbol 1, so that $A=1$ means that all the objects of the universe belong to the class A . The formula $A=0$, on the other hand, means that in the universe there are no objects of class A . In the same way as we have done before, we can attach to the $+$ and \cdot signs the meaning of «or» and «and». Therefore, $A+B$ means the operator which selects all objects which belong either to class A or to class B ; AB those who belong to both classes A and B at the same time. So if A means horses, and B white animals, to $A+B$ belong both horses and white animals, while to AB belong only the white horses.

The operator $1-A$ may be defined as the class of the objects of the universe which do not belong to class A . The following manipulation rules hold for any operator A, B :

$$(1) \quad AB = BA; \quad A^2 = A \cdot A = A; \quad A + A = A; \quad A(1-A) = 0.$$

In the algebra of classes so obtained, there is a general rule to solve a system of equations, when it is asked to express a class symbol A as a function of all other class symbols B, C, \dots . We multiply each equation, written in the form $f_n(A, B, C, \dots) = 0$, by a different Lagrangian multiplier λ_n and add all equations together. Let the result be:

$$(2) \quad F(A, B, C, \dots; \lambda_1, \lambda_2, \dots) = 0.$$

A is then given by the development

$$(3) \quad A = \sum_{i,k,l,\dots} \alpha(i, k, l, \dots; \lambda_1, \lambda_2, \dots) b_i c_k d_l \dots; \quad i, k, l, \dots = 0, 1.$$

where $b_0 = 1 - B$; $b_1 = B$; $c_0 = 1 - C$, $c_1 = C$ etc.. The $b_i c_k d_l \dots$'s form a complete set of orthogonal classes; i.e., all the objects of the universe belong to one and only one of these classes. The coefficients of the development are given by the following relation:

$$(4) \quad \alpha(i, k, l, \dots; \lambda_1, \lambda_2, \dots) = \frac{F(0, i, k, l, \dots; \lambda_1, \lambda_2, \dots)}{F(0, i, k, l, \dots; \lambda_1, \lambda_2, \dots) - F(1, i, k, l, \dots; \lambda_1, \lambda_2, \dots)}.$$

These coefficients are functions of the Lagrangian multipliers λ_n , which can be determined so that all coefficients take one of the following forms: 0/1; 1/1; 0/0; 1/0. Clearly, in dealing with Lagrangian multipliers one has to remember that they are classes and therefore obey to the rules (1) given above.

The interpretation of the coefficients is as follows:

$\frac{0}{1} = 0$ means that *no* element of the corresponding class is contained in class A , therefore this term does not appear in the development (3).

$\frac{1}{1} = 1$ means that *all* elements of the corresponding class belong to A .

$\frac{0}{0} =$ means that *some* elements of the corresponding class may belong to A ; we can therefore introduce a class symbol n (meaning «some») as coefficient in (3).

$\frac{1}{0} =$ means that the corresponding class has to be put equal to zero, and so gives rise to a separate condition.

For the proof (quite elementary) of this rule and any further detail on the subject of this section, the reader is referred to the quoted book of BOOLE⁽²⁾.

3. - An Example.

Take Problem 2 of Section 1. Let A_a be the statement « a belongs to tribe A »; B_a « a belongs to tribe B » and so on. We can immediately write down the following consistency equations:

$$(5) \quad A_n + B_n + C_n = 1, \quad n = a, b, c,$$

which means: «the statement “ n either belongs to tribe A or to tribe B or to tribe C ” is true»;

$$(6) \quad A_n B_n = A_n C_n = B_n C_n = 0, \quad n = a, b, c,$$

which means that statements of the type « n belongs to tribe A and to tribe B at the same time» are certainly wrong.

Let us now translate Table I into equations. We shall do so by small steps for better understanding. Let us consider the three answers of a : three cases are possible, according to whether a belongs to tribe A , B or C .

- 1) If a belongs to tribe A , then all the answers are wrong, so b is either B or C ; c is either B or C , i.e.:

$$\text{if } A_a = 1, \quad \text{then } A_a(B_b + C_b)(B_c + C_c) = 1.$$

- 2) If a belongs to tribe B , then all the answers are right; however, this is inconsistent with a saying that he is C . Therefore $B_a = 0$.

- 3) If a belongs to tribe C , then he has started by saying the truth; therefore b must belong either to tribe B or to tribe C , and c to tribe A , i.e.:

$$\text{if } C_a = 1, \quad \text{then } C_a(B_b + C_b)A_c = 1.$$

Since one of the alternatives is the truth, we can write:

$$(7) \quad A_a(B_b + C_b)(B_c + C_c) + C_a(B_b + C_b)A_c = 1 \quad \text{and} \quad B_a = 0.$$

Similarly, by writing down the statements of b and c , we get:

$$(8) \quad (A_a + C_a)B_b(B_c + C_c) + (A_a + C_a)C_b(B_c + C_c) = 1 \quad \text{and} \quad B_b = 0,$$

$$(9) \quad (B_a + C_a)(A_b + C_b)A_c + A_a(A_b + C_b)C_c = 1 \quad \text{and} \quad B_c = 0.$$

Since $B_a = B_b = B_c = 0$, the first equations of (7), (8), (9) become:

$$(7') \quad A_a C_b C_c + C_a C_b A_c = 1$$

$$(8') \quad A_a C_b C_c = 1,$$

$$(9') \quad A_a A_b C_c + A_a C_b C_c = 1.$$

Equation (8') gives immediately:

$$(10) \quad A_a = C_b = C_c = 1.$$

and from formulae (5) and (6):

$$(10') \quad A_b = A_c = B_a = B_b = B_c = C_a = 0.$$

It is easily seen that (10) satisfies equations (5)-(9) and is therefore the required solution. In plain words: a belongs to tribe A , both b and c belong to tribe C .

4. - A Physical Example.

We shall now consider the well known discussion by YANG⁽²⁾ of those systems which decay in two photons and translate it into the formalism of mathematical logic. Our «universe» includes only those systems which decay in two photons; in this universe we may define the following class operators:

A : systems with even parity;

B : systems in which the two photons are circularly polarized in the same way (LR and RL in Yang's notations);

C : systems of even total angular momentum (in \hbar units);

D : systems of total angular momentum 0;

E : systems of total angular momentum 1.

The following equations hold:

$$(11) \quad D(1 - C) = 0,$$

(2) C. N. YANG: *Phys. Rev.*, **77**, 242 (1950).

i.e. 0 is an even number (more precisely: there is no such system that its total angular momentum is 0 and not even at the same time):

$$(12) \quad EC = 0,$$

i.e. 1 is an odd number:

$$(13) \quad DB = 0,$$

$$(14) \quad EB = 0,$$

i.e. systems with total angular momentum 0 or 1 cannot decay in two photons polarized in the same way (since the component of the total angular momentum in the propagation direction would already be ± 2):

$$(15) \quad (1 - A)(1 - B) = (1 - A),$$

i.e. in all odd states the two photons are polarized in a different way:

$$(16) \quad (1 - C)(1 - B) = 0,$$

i.e. there is no system of odd total angular momentum in which the two photons are polarized in a different way (this being a consequence of the fact that odd spherical harmonics change sign for a rotation of 180° about an axis orthogonal to the quantization axis, whereas systems of two differently polarized photons are invariant for such rotation).

Equations (11)-(16) give all the physics of the problem.

Multiplying equation (16) by E we get:

$$E(1 - C)(1 - B) = 0,$$

which yields, taking into account eq. (14):

$$E(1 - C) = 0.$$

This equation, together with equation (12), gives identically:

$$(17) \quad E = 0.$$

This equation states that there exists no system of total angular momentum 1 which decays in two photons, as may be found in Table II of YANG's paper.

Taking into account eq. (17) and using the general method for the solution of systems of equations given at the end of Section 2, we obtain:

$$(18) \quad D = vA(1 - B)C + v'(1 - A)(1 - B)C.$$

This means that systems with total angular momentum 0 *may* (see the meaning of the v symbols) decay only in two photons differently polarized, no matter if the state is even or odd. This result, as well as any other result of YANG's paper, can be obtained in a similar fashion, by straightforward algebra.

5. — Discussion.

The example given above is quite elementary, and it may be questioned whether in this case the introduction of the formalism of mathematical logic is of any advantage at all. The important point is that with this method the solution is brought forward by completely automatic procedure, while with the usual method it has to be found by trial and error. This advantage becomes greater and greater as soon as the complication of the problem increases.

We have not been able so far to find any problem too difficult to be solved with usual methods. However, it seems to us a rather optimistic view to admit that physics is too simple! We should better believe that since physicists have for centuries considered all problems from the angle of usual mathematics, the whole development of physics has been conditioned by this outlook. A comparable training in mathematical logic would be necessary in order to make this kind of approach as familiar as that of usual mathematics. The ancient physicists were perhaps confronted with the same kind of difficulty when trying to translate for the first time the « common sense » concepts into mathematical symbols. Moreover, the elements of mathematical logic we summarized are possibly but a small part of a complete theory, such as arithmetics is in comparison with calculus.

We should eventually like to point out that the basic difference between mathematical logic and ordinary mathematics is not in the formalism, which is just one of the many formalisms that can be devised, but in the interpretation of the formalism. Therefore, to ask whether mathematical logic is of any use in physics, has a quite different meaning from the question of the relevance for physics of any other branch of mathematics. In this sense, the recent applications of Boolean algebra to the study of control and relais systems have nothing to do with mathematical logic.

RIASSUNTO

Si discute la possibilità di applicare la logica matematica alla risoluzione di problemi fisici. A tale scopo se ne introducono i concetti fondamentali, e si mostra come se ne possa utilizzare il formalismo.

Space and Time Reflection of Observable and Non-Observable Quantities in Field Theory.

G. MORPURGO and B. F. TOUSCHEK

*Scuola di Perfezionamento in Fisica Nucleare dell'Università - Roma
Istituto Nazionale di Fisica Nucleare - Sezione di Roma*

(ricevuto il 4 Aprile 1955)

Summary. — The definition of time reversal (given in a previous paper ⁽¹⁾) based only on the transformation properties of one particle observables is extended to the case of space reflection and put on a basis in which no preference is given to the momentum set of occupation numbers. This considerably restricts the class of the F -functions, with the effect that for a reversible system of fields the observables transform in a definite way while the nonobservables do not. The formalism is applied to the problem of Fermion types. It is shown that the exclusion of the interactions leading to the loss of heavy particles cannot be justified by an assignment of types to the Fermions.

Introduction.

In a previous paper ⁽¹⁾ we have given a formulation of time reversal which was based entirely on the consideration of observables; it was shown in I and elaborated in more detail in a second paper ⁽²⁾ that there exist systems of fields which can be unambiguously classified as irreversible on the basis of the proposed formulation. The purpose of the present paper is to complete and extend the results obtained in the first for systems of interacting fields. Though the definition of reversibility given in I ⁽³⁾ was independent of the particular

⁽¹⁾ G. MORPURGO, L. A. RADICATI and B. F. TOUSCHEK: *Nuovo Cimento*, **12**, 677 (1954) (I).

⁽²⁾ G. MORPURGO and B. F. TOUSCHEK: *Nuovo Cimento*, **1**, 201 (1955).

⁽³⁾ A system is reversible if there exists a unitary matrix K satisfying

$$KH^*K^+ = H \quad \text{and} \quad KN_Q^*K^+ = N_{\bar{Q}};$$

H is the Hamiltonian of the system, Q a set of oneparticle observables and N_Q the operator of the occupation number of a state labelled by Q . The Q transform into \bar{Q} under timereversal, \bar{Q} being defined by the correspondence principle.

set of one particle observables Q , which label the occupation numbers, we had in practice only used one set of observables: the set which contained the momentum and spin of the particles. As a consequence of this a large class of unitary matrices K had to be explored in order to decide whether a system was reversible or not and for reversible systems the transformation properties of the observables of the field were not unambiguously determined.

In the present paper we want to formulate more precisely and exploit more fully the definition of time reversal ⁽³⁾ given in I. We shall treat all the possible sets of one particle observables on the same footing not giving any preference to the momentum representation of the theory. We shall show that in this manner the arbitrariness of K is strongly restricted and that as a consequence the problem of deciding whether a system is reversible or not is greatly simplified and the transformation properties of field observables are uniquely determined.

We shall further show that the formalism and the conclusions on time reversal discussed in I and in this paper may also be extended to cover the operation of spatial reflection.

This enables us to consider the problem of Fermion «types». We will show, that on the basis of our formalism, based only on the transformation properties of observables it is not meaningful to attribute a type to a Fermion and then exclude certain interactions because, with the types of the Fermions fixed, they are not invariant under space or time reflection. We shall show in particular that in the frame of the theory of universal Fermi-interaction, the reaction

$$(1) \quad p \rightarrow \bar{n} + \nu + e^+$$

(where p is a proton, \bar{n} is an antineutron, ν a neutrino and e^+ an electron) cannot be prohibited by the attribution of a definite type to all the particles concerned. This is due to the fact that there exists a canonical transformation, having no effect on all the observable quantities, which leaves invariant the interaction Hamiltonians which conserve the nucleon number and which changes the sign of that part of the Hamiltonian which gives rise to reaction (1); a situation similar to the one discussed in I for the Hamiltonian $\frac{1}{2}p^2 + (\mathbf{p} \cdot \mathbf{r})V(r)$. It is therefore shown that the introduction of types in a theory of universal Fermi-interaction is not an adequate expedient to ensure the conservation of the number of heavy particles.

1. - The most general K satisfying the condition $\hat{N}_a(Q) = N_a(\hat{Q})$.

1.1. *The case of a scalar boson field.* - To illustrate our method we start by considering the case of a spinless neutral meson field. In I the matrix K

referring to such a field was subjected to the condition

$$(2) \quad KN_k^* K^+ = N_{-k}.$$

Here N_k is the occupation number of a state in which the momentum of the meson is k . It was shown, that if a matrix K_0 is defined in such a way, that

$$(3) \quad \hat{a}_k^+ = K_0 a_k^* K_0^+ = a_{-k}$$

(where the a_k , a_k^+ are respectively destruction and creation operators), the most general matrix K must be of the form

$$(4) \quad K = \exp [iF(N_k)] K_0,$$

where $F(N_k) = F(N_{k_1}, N_{k_2}, \dots)$ is an arbitrary Hermitian function of all the operators N_k . We now show that the class of F functions is considerably restricted by treating all the possible one meson observables on the same footing as the momentum variables.

We shall denote by Q a complete set of commuting observables for a single boson. One such set is represented by the kinetic momentum, another for example by the three components of the position vector of the meson. The behaviour under time reversal of the observables Q of a single particle is completely determined. For, every observable relating to a point spinless particle may be expressed as a function of position \mathbf{x} and velocity $\dot{\mathbf{x}}$, which transform according to

$$(5) \quad \hat{\mathbf{x}} = \mathfrak{f} \mathbf{x}^* \mathfrak{f}^+ = \mathbf{x}, \quad \hat{\dot{\mathbf{x}}} = \mathfrak{f} \dot{\mathbf{x}}^* \mathfrak{f}^+ = -\dot{\mathbf{x}}$$

under time reflection. \mathfrak{f} is a unitary matrix which acts only on the one particle variables.

Now let ψ_a be a set of normalised eigenstates of Q :

$$(6) \quad Q\psi_a = q\psi_a.$$

With any choice of \mathfrak{f} which satisfies (5) we have

$$(7) \quad \hat{Q}\hat{\psi}_a = q\hat{\psi}_a$$

where

$$(8) \quad \hat{\psi}_a = \mathfrak{f} \psi_a^*.$$

This means that if ψ_a is an eigenfunction of Q to the eigenvalue q , then $\hat{\psi}_a$ is an eigenfunction of \hat{Q} to the same eigenvalue.

Let us now denote by $N_q(Q)$ the operator of the occupation number of a one particle eigenstate ψ_q of the operator Q . Then $N_q(\hat{Q})$ will be the occupation number operator of the eigenstate $\eta_q \hat{\psi}_q$ of the operator \hat{Q} . η_q is a phase factor which as we shall see will cancel out in the following considerations. Before discussing the way in which the operators $N_q(Q)$ and $N_q(\hat{Q})$ can be actually constructed, we observe that the general requirement of reversibility will now be

$$(9) \quad \hat{N}_q(Q) = K N_q^*(Q) K^+ = N_q(\hat{Q}),$$

for all sets Q of one particle observables. This is a more precise formulation of eq. (I.78).

The $N_q(Q)$ can be constructed in the following manner. Let φ_k be the normalised eigenfunction (in a given representation) of the one particle momentum operator G :

$$(10) \quad G\varphi_k = k\varphi_k.$$

By a suitable choice of the phasefactors in \mathfrak{f} and φ_k we may arrange that ⁽⁴⁾

$$(11) \quad \hat{\varphi}_k = \mathfrak{f}\varphi_k^* = \varphi_{-k}.$$

Creation operators $a_q^+(Q)$ and destruction operators $a_q(Q)$ may now be introduced by

$$(12) \quad a_q^+(Q) = \sum_k a_k^+(\varphi_k^* \psi_q), \quad a_q(Q) = \sum_k (\psi_q^* \varphi_k) a_k,$$

where a_k^+ and a_k are the creation and destruction operators in the momentum representation of the field. (12) insures that the 1-particle states in second quantisation transform in the same way as the states of a one particle theory. The $a_q^+(Q)$, $a_q(Q)$ satisfy the same commutation relations as the a_k^+ , a_k . The occupation numbers $N_q(Q)$ are then given by

$$(13) \quad N_q(Q) = a_q^+(Q) a_q(Q) = \sum_{kk'} a_k^+ a_{k'} (\varphi_k^* \psi_q) (\psi_q^* \varphi_{k'})$$

from which it follows that

$$(14) \quad N_q(\hat{Q}) = \sum_{kk'} a_k^+ a_{k'} (\varphi_k^* \hat{\psi}_q) (\hat{\psi}_q^* \varphi_{k'}),$$

⁽⁴⁾ The situation is more complicated in the general case and will be discussed in section 1.3.

because the most general eigenstate of Q to eigenvalue q may differ from $\hat{\psi}_q$ at most by a phasefactor.

1.2. *Consistency and most general solution of eq. (9).* — In this section we want to show two properties of the set of equations (9) in the case of spinless neutral mesons.

1) The set of equations is internally consistent. The matrix K_0 defined by equation (3) automatically satisfies equations (9) for any set of oneparticle observables Q and for any q .

2) The most general matrix K which satisfies (9) is related to the matrix K_0 by the equation

$$(15) \quad K = \exp \left[iF \left(\sum_k N_k \right) \right] K_0,$$

in which F is an arbitrary Hermitian function of just the sum of all the meson occupation numbers.

To show the first point we observe that with K_0 defined by (3) we have

$$(16) \quad K_0 N_a^+ K_0^+ = \sum_{kk'} a_{-k}^+ a_{-k'} (\varphi_a^* \psi_k) (\varphi_a^* \psi_{k'}).$$

It follows from equation (11), that

$$(17) \quad (\varphi_{-k} \psi_a^*) = (\hat{\varphi}_k \psi_a^*) = (\varphi_k^* \hat{\psi}_a) \quad (\text{any } k \text{ and } q)$$

the last passage being possible for the fact that here $\mathfrak{f} = \mathfrak{f}^T$; notice that a possible choice for \mathfrak{f} in this case is $\mathfrak{f} = I$ the «unit» matrix.

Comparing now (16) with equation (14) we see with the help of (17) that equation (9) is satisfied. The equivalent of equation (17) will be demonstrated for the general case in section 1.3.

To show the validity of equation (15) we remember that the most general matrix K which satisfies (2) was given by equation (4). In order that (9) be satisfied we must have

$$(18) \quad \exp [iF(N_{k_1}, N_{k_2}, \dots)] \cdot \sum_{kk'} a_k^+ a_{k'} (\varphi_k^* \hat{\psi}_q) (\hat{\psi}_a^* \varphi_{k'}) \cdot \exp [-iF(N_{k_1}, N_{k_2}, \dots)] = \\ = \sum_{kk'} a_k^+ a_{k'} (\varphi_k^* \hat{\psi}_q) (\hat{\psi}_a^* \varphi_{k'}).$$

Let us now take the special set of observables which represent the position \mathbf{x} of the mesons. Then

$$(19) \quad (\varphi_k^* \hat{\psi}_x) = \exp [-i\mathbf{k} \cdot \mathbf{x}].$$

Since because of (19) $(\varphi_k^* \hat{\psi}_x) \neq 0$ for any value of k it follows that

$$(20) \quad \exp [iF(N_{k_1}, N_{k_2}, \dots)] a_k^+ a_{k'} \exp [-iF(N_{k_1}, N_{k_2}, \dots)] = a_k^+ a_{k'}$$

must hold. Taking the matrixelement of this equation between the states $\langle \dots N_k + 1 \dots N_{k'} \dots |$ and $| \dots N_k \dots N_{k'} + 1 \dots \rangle$ it follows that

$$(21) \quad F(\dots N_k + 1 \dots N_{k'}) = F(\dots N_k \dots N_{k'} + 1 \dots) \pmod{2\pi}.$$

Since this equation must hold for every pair k and k' it follows easily that F can only depend on the sum of all the N_k .

An essential point of the proof of equation (15) is that it should be possible to choose a set of observables Q in such a way that for a given pair of values k and k' there exists an eigenvalue q for which both $(\hat{\psi}_q^* \varphi_{k'})$ and $(\varphi_k^* \hat{\psi}_q)$ do not vanish. This condition is satisfied as we have seen for neutral spinless mesons, but has to be discussed in more detail for more complicated systems. This will be done in section 1·4.

The limitation on the admissible choices of F effected by the generalisation (9) of (2) is obvious from a comparison of (15) and (4).

1·3. *Extension to more general fields.* — The results obtained in the special case of a spinless neutral mesonfield can be extended to the case of any field. We want to show that the first theorem (of the previous section) holds quite generally, whereas the second needs some qualifications.

In the following we shall refer to the set of observables discussed in I as the momentum set — though this set may include other variables as well. In the case of Dirac particles it comprises momentum \mathbf{G} , spin in the direction of the momentum ($\mathbf{G} \cdot \boldsymbol{\sigma}$) and a projection operator \mathcal{A} , which distinguishes between particles and antiparticles. In the case of charged mesons we shall have an observable which distinguishes the various charge states in addition to the momentum. We shall denote the set of momentum observables by P and its eigenvalues by p . The eigenfunctions of the momentum set will be denoted by φ_p , thus

$$(22) \quad P\varphi_p = p\varphi_p.$$

Under the transformation of time reversal effected by a matrix \mathfrak{t} of the one particle problem, chosen in such a way that every member of the set P has the classical transformation properties, any particular member of the set P will change sign or not. We may write:

$$(23) \quad \hat{P} = \mathfrak{t}P^*\mathfrak{t}^\dagger = \bar{P}.$$

Equation (23) may be inverted giving

$$P = \bar{\hat{P}} = \hat{\bar{P}}.$$

With $\hat{\varphi}_p = \mathfrak{k} \varphi_p^*$ we have $P \hat{\varphi}_p = p \hat{\varphi}_p$. But because of the last equation $\hat{\varphi}_p$ is also eigenfunction of the operator P belonging to the eigenvalue \bar{p} . We may therefore write

$$(24) \quad \hat{\varphi}_p = \gamma_p^* \varphi_{\bar{p}},$$

where γ_p is a phasefactor. In general it is not possible to make $\gamma_p = 1$, as in the case of spinless mesons. In the case of Dirac particles for example γ has to satisfy $\gamma_p + \gamma_{\bar{p}} = 0$ ⁽⁵⁾.

For a general set of one particle observables Q we may introduce creation and annihilation operators by means of the relations

$$(25) \quad a_a^+(Q) = \sum_p a_p^+(\varphi_p^* \psi_a), \quad a_a(Q) = \sum_p (\psi_a^* \varphi_p) a_p.$$

The general postulate of reversibility we retain in the form (9). The first theorem of section 1.2 may now immediately be verified. We claim that the transformation

$$(26) \quad \hat{a}_p^+ = \gamma_p a_{\bar{p}}, \quad \hat{a}_p = \gamma_p^* a_{\bar{p}}^+,$$

defines a K_0 which satisfies equation (9). The introduction of the factor γ_p does not change the commutation relations between the \hat{a} , \hat{a}^+ since γ is unimodular.

Using equation (26) we find

$$(27) \quad K_0 N_a^*(Q) K_0^+ = \sum_{p, p'} a_p^+ a_{p'} (\varphi_{\bar{p}} \varphi_{\bar{p}'}^*) (\psi_a \varphi_{\bar{p}'}^*) \gamma_{\bar{p}}^* \gamma_{\bar{p}'}.$$

⁽⁵⁾ Notice that for the case of Dirac particles equation (24) may be written as

$$u(-k, s, \varepsilon) = \gamma(-k, s, \varepsilon) U u^*(k, s, \varepsilon)$$

in the notation of (I). U corresponds to \mathfrak{k} and u to φ of the present paper. Since for any choice of the phasefactor η in $U = \eta \alpha_1 \alpha_2 \alpha_3 \beta$ one has $U U^* = -1$ it follows that in the case of Dirac particles one must have

$$\gamma(-k, s, \varepsilon) + \gamma(k, s, \varepsilon) = 0.$$

Equations (I.74) as well as (I.71) are accordingly incorrect, though the result (I.75) still holds.

On the other hand we have

$$(28) \quad N_q(\hat{Q}) = \sum_{pp'} a_p^+ a_{p'} (\varphi_p^* \hat{\psi}_q) (\hat{\psi}_q^* \varphi_{p'}) .$$

But

$$(29) \quad (\varphi_p^* \hat{\psi}_q) = (\psi_q^* \mathfrak{f} \varphi_p^*) = \alpha (\psi_q^* \hat{\varphi}_p) = \alpha \gamma_p^* (\psi_q^* \varphi_{p'}^-) ,$$

where α is a constant equal to ± 1 . The reason is that because of the relation $\hat{Q} = Q$ for every one particle observable, $\mathfrak{f}\mathfrak{f}^*$ must commute with all the observables of the one particle system. (It follows that $\mathfrak{f}\mathfrak{f}^*$ must be a real multiple of the unit matrix and because of $\mathfrak{f}\mathfrak{f}^+ = 1$ must be ± 1). The choice (25) of K_0 therefore ensures that (9) is satisfied for any set of one particle observables.

For the fields which we shall consider in the following \mathfrak{f} and γ_p may be chosen in such a way, that their transformation properties are the orthodox ones and identical with those given in I. Namely:

- 1) For a Dirac field:

$$K_{0d} \psi^*(\mathbf{r}) K_{0d}^+ \equiv \hat{\psi}^+(\mathbf{r}) = U \psi(\mathbf{r})$$

with

$$U = i\alpha_1 \alpha_2 \alpha_3 \beta .$$

- 2) For a charged spinless Boson field

$$K_{0m} \Phi^*(\mathbf{r}) K_{0m}^+ \equiv \hat{\Phi}^+(\mathbf{r}) = - \Phi(\mathbf{r}) .$$

- 3) For a neutral spinless Boson field:

$$K_{0m} \Phi^*(\mathbf{r}) K_{0m}^+ \equiv \hat{\Phi}(\mathbf{r}) = - \Phi(\mathbf{r}) .$$

- 4) For the transverse part of the electromagnetic field:

$$K_{0\gamma} \mathbf{A}^*(\mathbf{r}) K_{0\gamma}^+ \equiv \hat{\mathbf{A}}(\mathbf{r}) = - \mathbf{A}(\mathbf{r}) .$$

1.4. *One particle observables and the properties of F.* — To extend theorem 2) of section 1.2 to the general case some qualifications are necessary. As we have already observed in that section the proof of 2) was based on the fact that it was possible to find a set of observables Q such that for every pair of eigenvalues k, k' there existed at least one eigenvalue q for which both $(\varphi_k^* \hat{\psi}_q)$ and $(\hat{\psi}_q^* \varphi_{k'})$ were different from zero. In the general case, we would have to require the same property of the products $(\varphi_p^* \hat{\psi}_q)$ and $(\hat{\psi}_q^* \varphi_{p'})$. It is

easy to convince oneself, that this amounts to saying that for every pair of eigenfunctions φ_p and $\varphi_{p'}$ there exists at least one observable Q such that

$$(30) \quad (\varphi_p^* Q \varphi_{p'}) \neq 0.$$

If (30) is satisfied we may proceed as in section 1.2 and F will depend only on the sum of all the occupation numbers. Now consider as an example a charged meson field. In this case p stands for the momentum K and a charge index capable of the values $+$ and $-$. It is a well known fact ⁽⁶⁾ that no meaning can be given to the relative phase of a superposition of two one particle states corresponding to different values of the charge. This means that there can not exist an observable with non vanishing matrix elements between two one particle states belonging to a different charge. Hence $(\varphi_p^* Q \varphi_{p'})$ must necessarily vanish if p and p' differ by the charge index i . The argument which leads to equation (15) must therefore be modified ⁽⁷⁾. The result is that the most general K -matrix for the charged Boson field, which satisfies condition (9) is obtained from K_0 by multiplication with $\exp[iF(\sum_k N_{k+}, \sum_k N_{k-})]$ where F is a now a Hermitian function of the two arguments $\sum_k N_{k+}$ and $\sum_k N_{k-}$.

The situation is quite similar in the case of a Dirac field. PRICE and FOLDY and WOYTHUSEN ⁽⁸⁾ have shown that the observables of the Dirac field — i.e. the quantities which have a physical significance in the correspondence limit — are such that their matrix elements between states of positive and negative energy vanish. A set of observables which has this property consists of what FOLDY and WOYTHUSEN call the « mean position » and the « mean spin angular momentum » and a charge (or particle-antiparticle) index ⁽⁹⁾. As a consequence the most general K of the Dirac field will be $K = \exp[iF(\sum_{ks} N_{ks+}, \sum_{ks} N_{ks-})] K_0$. It is important to note the essential difference between the charge and spin index: there are observables which change the spin, but there are none which change the charge.

There is no such selection principle in the case of photons. The most general K is of the form $K = \exp[iF(\sum_{ks} N_{ks})] K_0$.

Summing up the situation we may say that the most general function F for a system of non-interacting photons, charged and neutral mesons, and Dirac particles (for which a distinction can be made between particles and anti-

⁽⁶⁾ L. L. FOLDY: *Phys. Rev.*, **93**, 1395 (1954).

⁽⁷⁾ It is no more true that necessarily $F(\dots N_p + 1 \dots N_{p'} \dots) = F(\dots N_p \dots N_{p'} - 1 \dots)$ if p and p' belong to different values of the charge index.

⁽⁸⁾ M. H. L. PRICE: *Proc. Roy. Soc. A* **195**, 62 (1948); L. L. FOLDY and S. A. WOYTHUSEN: *Phys. Rev.*, **78**, 29 (1950).

⁽⁹⁾ In this sense the set $G, (\sigma \cdot G), A$ used in I are true observables.

particles) will be of the form

$$(31) \quad F = F(n_m^0, n_m^+, n_m^-, n_f, n_f^+, n_f^-).$$

Here n_m^0 is the total number of neutral mesons, n_m^+ , n_m^- the number of positive or negative mesons, n_f^+ the total number of fermions f , and n_f^- the total number of antifermions.

2. - Systems of Interacting Fields.

2.1. Generalities. - We have shown in the previous section that for a system of non interacting fields, the freedom in the choice of F is considerably restricted by the postulate expressed by equation (9). The most general F was given by equation (31). The restriction on the functions F expressed by (31) facilitates greatly the decision on whether or not a given system of interacting fields is reversible. It is also possible to show that for a reversible system of fields the transformation properties of all field observables are uniquely determined by the extended postulate (9); the same is not true for the transformation of those field operators, which are not observables.

We do not intend to prove these statements in their full generality but rather confine our attention to the treatment of a standard system of nucleons, charged and neutral mesons and photons, showing that in this case our statements are true.

2.2. Simple treatment of an irreversible system. - Let us first revise the proof of the irreversibility of a system of scalar neutral mesons coupled with scalar and vector coupling to a nucleon field. We shall show that with the new postulate (9) this proof is straightforward.

With the same notation as in I, we may write for the interaction

$$(32) \quad H_1 = \mathcal{H}_1 + \mathcal{H}_2$$

which under the transformation generated by $K_0 = K_{0m} \cdot K_{0a}$ goes into ⁽¹⁰⁾

$$(33) \quad \hat{H}_1 = K_0 H_1^* K_0^+ = \mathcal{H}_2 - \mathcal{H}_1.$$

Recalling (I.85) and (I.86) we have to show that there is no choice of F which makes

$$(34) \quad \exp [i(F(\mathcal{Q}'') - F(\mathcal{Q}'))] = \frac{\langle \mathcal{Q}'' | \mathcal{H}_1 + \mathcal{H}_2 | \mathcal{Q}' \rangle}{\langle \mathcal{Q}'' | \mathcal{H}_2 - \mathcal{H}_1 | \mathcal{Q}' \rangle}.$$

⁽¹⁰⁾ The difference between (34) and (I.85) ($\hat{H}_1 = \mathcal{H}_1 - \mathcal{H}_2$) is due to the fact that K_{0m} used here transforms Φ into $-\Phi$ while for the one used in I we had $\hat{\Phi} = \Phi$.

FONDAZIONE FRANCESCO SOMAINI PRESSO IL TEMPIO VOLTIANO, A COMO

BANDO DI CONCORSI AL PREMIO E ALLA BORSA PER IL 1955

Con lo scopo di premiare e incoraggiare nel nome di ALESSANDRO VOLTA gli studi di Fisica in Italia, la « Fondazione Francesco Somaini » presso il Tempio Voltiano a Como, indice i seguenti Concorsi:

- A) CONCORSO AL « PREMIO TRIENNALE PER LA FISICA FRANCESCO SOMAINI » PER IL 1955 di L. 1500000 (un milione e cinquecentomila) nette, da assegnarsi al concorrente che, fra quelli che la Commissione Giudicatrice giudicherà in senso assoluto meritevoli del Premio per i risultati conseguiti nello studio della Fisica durante il Triennio 1° Luglio 1952, 30 Giugno 1955, sia, a parere della Commissione stessa, il più meritevole.
- B) CONCORSO ALLA « BORSA FRANCESCO SOMAINI PER LO STUDIO DELLA FISICA » PER IL 1955 di L. 750000 (settecentocinquantomila) nette, da assegnarsi al concorrente che, fra quelli che la Commissione Giudicatrice giudicherà in senso assoluto meritevoli della Borsa, verrà dalla Commissione stessa giudicato il più meritevole, sia per titoli, preparazione scientifica, lavori già svolti e risultati già conseguiti, nella Fisica, sia anche per il vantaggio che gli studi, per i quali è richiesta la Borsa, possono portare allo sviluppo della Fisica in Italia.

1. — Ad entrambi i Concorsi possono prendere parte singolarmente i cittadini d'ambo i sessi italiani e svizzeri del Canton Ticino purchè di stirpe italiana. Sono esclusi dal Concorso i membri della Commissione Amministratrice e della Commissione Scientifica della « Fondazione Francesco Somaini ».

2. — Le norme particolareggiate dei singoli Concorsi verranno pubblicate in apposito volantino che potrà essere richiesto dagli interessati alla Segreteria della Fondazione presso il Tempio Voltiano a Como.

3. — La domanda, i documenti, i lavori, ecc., presentati dai singoli concorrenti dovranno pervenire, tra il 1° Gennaio e le ore 12 del 1° Luglio 1955, alla Commissione Amministratrice della « Fondazione Francesco Somaini » a Como presso il Tempio Voltiano.

La procedura dei suddetti Concorsi è regolata secondo lo Statuto della Fondazione il quale è ostensibile a Como presso il Tempio Voltiano ed è depositato presso il Notaio Dr. Raoul Luzzani di Como.

Como, dal Tempio Voltiano

Il giorno 26 Settembre 1952

Il Segretario

Il Presidente

Conservatore del Tempio

Sindaco di Como

CESARE MORLACCHI

GIUSEPPE TERRAGNI

ENTE NAZIONALE IDROCARBURI

ROMA - Via Lombardia, 43

ASSUNZIONE DI PERSONALE DIRIGENTE

per laboratori di ricerca nell'industria petrolifera

L'Ente Nazionale Idrocarburi (E.N.I.) ha in corso di organizzazione un grande laboratorio per le ricerche nel campo delle industrie del petrolio e della petrochimica che entrerà in funzione con i suoi primi reparti verso l'estate prossima.

Per la direzione dei principali reparti del laboratorio si cercano studiosi, italiani o stranieri, di vasta cultura generale e di profonda preparazione con qualità organizzative e direttive.

In particolare si cercano:

- 1) chimico o fisico con preparazione indirizzata ai problemi di termodinamica, di elettrochimica e di chimica-fisica;
- 2) chimico versato nella preparazione di composti organici e in particolare in quelli ottenibili dagli idrocarburi (petrochimica);
- 3) chimico inorganico specializzato nei metodi moderni di preparazione, di ricerca e di controllo;
- 4) ingegnere o chimico industriale preparato nel campo degli studi per il perfezionamento delle caratteristiche di impiego dei prodotti petroliferi;
- 5) ingegnere con preparazione fisico-tecnica e tecnologica con particolare riguardo a problemi di termotecnica, di misure e di controlli;
- 6) ingegnere o chimico industriale con larga pratica di ricerca nel campo dei trattamenti di raffinazione degli idrocarburi greggi;
- 7) ingegnere o chimico industriale con pratica di realizzazione in fase semindustriale di procedimenti sviluppati in laboratorio (impianti pilota);
- 8) ingegnere specializzato nello studio teorico e nella sperimentazione su motori dei carburanti e dei lubrificanti;
- 9) ingegnere o chimico industriale preparato dal punto di vista teorico e pratico nel campo delle corrosioni e della protezione dei materiali.

Gli interessati sono invitati a trasmettere il loro curriculum vitae — corredato di referenze e di fotografia — indirizzandolo personalmente al *Presidente dell'Ente Nazionale Idrocarburi, Via Lombardia, 43, Roma*, con la soprascritta **RISERVATA PER SONALE**.

Si assicura il più assoluto riserbo sotto ogni aspetto.

Il trattamento economico da convenire sarà adeguato alla importanza dell'incarico e alla preparazione e alle qualità richieste.

Here, according to section 1.4, F is just a function of n_m^0 , n_f^+ , n_f^- . The left hand side is therefore independent of the momentum of the boson or fermion which effects the « transition » $\mathcal{Q}' \leftrightarrow \mathcal{Q}''$. Since the right hand side depends on the momentum of the Fermion which makes the transition, (34) cannot be satisfied and the system is therefore irreversible.

The previous argument may be extended considerably. Let H_1 be the interaction between two or more fields and assume that H_1 is the sum of two terms \mathcal{H}_1 and \mathcal{H}_2 which have nonvanishing matrixelements between the same states. Assume also that under a particular transformation of time reversal induced by a matrix K_0 , \mathcal{H}_1 and \mathcal{H}_2 are multiplied by different phasefactors. Then the system is irreversible. This justifies the standard procedure of diagnosing time irreversibility of a system of this type.

2.3. A reversible system. Definite transformation properties of field observables. — In this section we want to examine a typical reversible system made up of interacting nucleons, mesons and photons. We shall assume that the interaction between nucleons and mesons is charge symmetric and pseudo-scalar, viz.

$$(35) \quad H_{nm} = \int \{ g\sqrt{2}(\psi_n^+ \Gamma \psi_p \Phi^+ + \psi_p^+ \Gamma \psi_n \Phi) - g(\psi_n^+ \Gamma \psi_n - \psi_p^+ \Gamma \psi_p) \Phi_3 \} d\tau,$$

with

$$(36) \quad \Gamma = \alpha_1 \alpha_2 \alpha_3 \beta.$$

The electromagnetic interaction of the mesons is given by

$$(37) \quad H_{\gamma m} = \int \{ -ie\mathbf{A} \cdot (\Phi^+ \nabla \Phi - \Phi \nabla \Phi^+) + e^2 A^2 \Phi^+ \Phi \} d\tau$$

and the electromagnetic interaction of the nucleons by

$$(38) \quad H_{\gamma n} = \int \{ -e\psi_p^+ \boldsymbol{\alpha} \psi_p \cdot \mathbf{A} \} d\tau.$$

Here ψ_p , ψ_n are the operators of protons and neutrons, Φ^+ , Φ is the charged meson field, Φ_3 describes the neutral mesons and \mathbf{A} is the transverse part of the electromagnetic four-potential.

Under the transformation induced by $K_0 = K_{0f} K_{0m} K_{0\gamma}$, defined in section 1.3 we have

$$(39) \quad \hat{H}_1 = \hat{H}_{nm} + \hat{H}_{\gamma m} + \hat{H}_{\gamma n} = H_{nm} + H_{\gamma m} + H_{\gamma n} = H_1$$

so the system is certainly reversible. We want to find the most general $K = \exp [iF] \cdot K_0$ where $F = F(n_p^+, n_m^+, n_m^0, n_p^-, n_p^-, n_n^+, n_n^-)$ which leaves H_1 invariant under time reversal. The result of a more detailed consideration of the possible matrixelements of H_1 given in the appendix is:

$$(40) \quad F = F(Q, N_h),$$

where Q is the total charge of the system

$$(41) \quad Q = n_p^+ - n_p^- + n_m^+ - n_m^-$$

and N_h the nucleon number:

$$(42) \quad N_h = n_p^+ - n_p^- + n_n^+ - n_n^-.$$

We see that the freedom of the choice of F is considerably restricted by the introduction of interaction: F now depends on only two numbers (allowed to assume values both positive and negative) instead of 8 positive numbers.

It follows from equations (40) and (15) that the most general choice of K which satisfies (9) and transforms H into itself is given by

$$(43) \quad K = \exp [iF(N_h, Q)] K_0.$$

From this it follows that the transformation properties of an arbitrary field observable are uniquely determined. Let us assume that the system described by the interactions (35), (37) and (38) is such that all measurements which can be performed on this system conserve its charge and the nucleon number⁽¹¹⁾. Then if Ω is an observable of the system constructed from the operators ψ , ψ^+ , etc., it must of course commute with the nucleon number N and with the charge Q . It follows that it must commute also with F (since $\hat{F} = F$), whatever choice of K is made. Therefore

$$(44) \quad \hat{\Omega} = \exp [(iF)] K_0 \Omega^* K_0^+ \exp [-iF] = K_0 \Omega^* K_0^+.$$

$\hat{\Omega}$ is therefore uniquely determined. The observables of the system transform in the manner defined by K_0 . It also follows that

$$(45) \quad \hat{\hat{\Omega}} = \Omega.$$

⁽¹¹⁾ If this were not the case and if for example the nucleon number could be altered by a measurement, the interaction which would lead to this alteration would have to be included in the list of interaction Hamiltonians and F could no longer depend on N_h .

This is of course not true of all the field operators, since these in general need not be observables.

Notice finally that if instead of the I -interaction we had assumed a γ_4 -interaction between nucleons and mesons, the conclusions at which we have arrived in this section would still be valid provided that for K_0 one chooses

$$(46) \quad K_0 = K_{0a} K_{0m} K_{0p} \exp [i\pi(n_m^+ - n_m^- + n_m^0)],$$

which is possible by virtue of equation (31). Here the exponent can of course not be expressed as a function of charge and nucleon number only. This is not a contradiction, since (43) is valid only if the Hamiltonian is invariant under K_0 . With the definition of K_0 given in equation (46) the most general K is again given by equation (43).

3. - Extension of the Formalism to Space Reflections.

The formalism developed so far for the case of time reflections can be easily extended to cover spatial reflections. Any one particle observable Q transforms in a definite way under space reflection, i.e. under a canonical transformation induced by a unitary operator, which transforms \mathbf{x} into $-\mathbf{x}$, $\dot{\mathbf{x}}$ into $-\dot{\mathbf{x}}$ and $\boldsymbol{\sigma}$ into $\boldsymbol{\sigma}$. Let \tilde{Q} be the result of the parity operation on Q . Then

$$(47) \quad \tilde{Q} = \mathfrak{f}' Q \mathfrak{f}'^+,$$

where \mathfrak{f}' is a unitary operator, operating on the one particle system.

A system of fields may now be called invariant with respect to reflections of space, if there exists an operator K' such that

$$(48) \quad \tilde{N}_a(Q) = K' N_a(Q) K'^+ = N_a(\tilde{Q})$$

and at the same time

$$(49) \quad K' H K'^+ = H,$$

H being the Hamiltonian of the system. This definition is quite similar to that given for time reversibility (eq. (19)) the only difference being that in the case of spacial reflections K' is an operator and that the transformations induced by K' are canonical and not anticanonical. The consequences of this definition are also quite similar to those drawn from time reversal. Denoting the reflected operator of a given operator O by \tilde{O} , we have

$$(50) \quad \tilde{O} = K' O K'^+.$$

In the case of the systems discussed in section 1.3 we can find operators $K'_{0\gamma}$, K'_{0m} , K'_{0d} , such that equations (48) and (49) are satisfied and which transform the field operators according to

$$(51) \quad \begin{cases} \tilde{A}(\mathbf{r}) = K'_{0\gamma} A(\mathbf{r}) K'^{+}_{0\gamma} = -A(-\mathbf{r}), \\ \tilde{\Phi}(\mathbf{r}) = K'_{0m} \Phi(\mathbf{r}) K'^{+}_{0m} = -\Phi(-\mathbf{r}), \\ \tilde{\psi}(\mathbf{r}) = K'_{0d} \psi(\mathbf{r}) K'^{+}_{0d} = i\beta\psi(-\mathbf{r}). \end{cases}$$

Also in the case of space reflections the most general K' is obtained from a given K' by multiplication by a function $\exp[iF']$ where F' depends on the same arguments as the function F defined in equation (31) for the case of time inversion.

The system of interacting fields discussed in section 2.3 is reflection invariant under the transformation effected by $K'_0 = K'_{0\gamma} K'_{0m} K'_{0d}$ and it is easily seen that in this case the most general parity transformation is given by

$$(52) \quad K' = \exp[iF'(N, Q)] K'_0.$$

An argument completely identical to the one discussed in the previous section then shows that the field observables transform in a definite manner, i.e. their transformation properties are independent of the particular choice of F' , but the field operators themselves do not.

4. - The Question of Fermion Types.

4.1. *Statement of the problem.* - In this section we want to apply the formalism of improper transformations based entirely on the transformation properties of observable quantities to the discussion of Fermion types. This involves considering also the other « elementary » particles as electrons, μ -mesons and neutrinos.

The fact that assuming a quadrilinear interaction between four Fermion fields, the value of the interaction constant came out to be the same (within the experimental error) for the three processes

$$(53) \quad \begin{cases} p \rightarrow n + e^+ + \nu \\ \mu \rightarrow e + \nu + \nu \\ p + \mu^- \rightarrow n + \nu \end{cases}$$

led YANG and TIOMNO to hypothesize a universal quadrilinear interaction between any four Fermion fields. This interaction therefore was assumed to

have the form

$$(54) \quad H_u = \int d\tau \sum_{rstu} \sum_{\alpha\beta\gamma\delta} \psi_\alpha^{(r)} \psi_\beta^{(s)} \psi_\gamma^{(t)} \psi_\delta^{(u)} A_{\alpha\beta\gamma\delta},$$

where $\alpha\beta\gamma\delta$ are spinor indices, $A_{\alpha\beta\gamma\delta}$ are numerical coefficients chosen in such a way as to give rise to a Lorentz-invariant interaction, r, s, t, u , are indices specifying the kind of Fermion, treating particles and antiparticles as different fields. The interaction (54) is called universal because A does not depend on the indices r, s, t, u . The sum over these indices covers all their combinations subject to the conservation of charge; i.e. those combinations of the indices r, s, t, u , which would lead to a violation of the conservation of charge in a process made possible by the interaction (54) are not included in the sum.

Having once assumed the universal interaction (54) YANG and TIOMNO were forced to give an explanation for the fact that not all the processes consistent with the conservation of charge and compatible with the interaction (54) actually take place in nature. In order to find a mechanism to forbid these unwanted interactions, YANG and TIOMNO⁽¹²⁾ examined the invariance properties of (54) under space reflections. They observed that the transformation properties of a spinor under reflections left a phasefactor undetermined. Assuming that there were four ways of transforming such a spinor (corresponding respectively to the values $\pm 1, \pm i$ of the phasefactor) they suggested that there should exist different «types» of Fermions — differing by the phasefactor in space reflection. Observing now that the type of the antiparticle is determined by the type of the particle, but that the relation between particle and antiparticle types is different for various types of particles, YANG and TIOMNO could show, that by a suitable choice of the types of all the particles concerned in the universal interaction the greater part of the unwanted reactions could be forbidden, since the interaction term leading to the unwanted interaction would not be invariant under space reflections. It was later shown by GAMBA⁽¹³⁾, that considering time reversal together with space reflections it is possible — and in more than one way — to exclude *all* the unwanted interactions by a suitable choice of types for space and time reflection.

We want to show, that on the basis of what has been established in this paper these conclusions are only partly correct. In particular we want to demonstrate that «typelike» considerations can not exclude the process (1)

$$p \rightarrow \bar{n} + e^+ + \nu$$

which is certainly unwanted.

⁽¹²⁾ C. N. YANG and J. TIOMNO: *Phys. Rev.*, **79**, 495 (1950).

⁽¹³⁾ A. GAMBA: *Nuovo Cimento*, **7**, 919 (1950). Compare also the papers quoted in I.

4.2. *Admissible and inadmissible interactions.* — We consider a general system of interacting nucleons, π -mesons, μ -mesons, electrons, neutrinos and photons. The interactions between these particles, which either have been observed or of which we believe that they might be observed in the future, we shall call «admissible interactions» and denote them by H_a . Among the admissible interactions a part will be due to the universal Fermi-interaction and we shall call this part H_{a_1} . Thus

$$(55) \quad H_a = H_{a_1} + H_{a_2},$$

H_{a_1} therefore includes all the interactions describing β -decay, μ -decay and μ -capture; H_{a_2} includes the interactions considered in section 2.3, the electromagnetic interactions of electrons and μ -mesons, the interaction which leads to the decay of the π -meson and all the other possible interactions, which are not quadrilinear interactions between 4 Fermions.

We shall call «inadmissible» the interactions which we want to exclude and we shall in particular fix our attention to inadmissible quadrilinear interactions between 4 Fermions which we shall denote by H_{in} . We may then put

$$(56) \quad H_{in} = \sum_k H_{in}^{(k)}.$$

Here the index k stands for those values of the 4 indices r, s, t, u , of equation (54) which would lead to inadmissible interactions. The interaction terms $H_{in}^{(k)}$ we shall call orthogonal, meaning that for $k \neq k'$ there does not exist a pair of states $\langle \mathcal{Q}' |$ and $| \mathcal{Q}'' \rangle$ such that both $\langle \mathcal{Q}' | H_{in}^{(k)} | \mathcal{Q}'' \rangle$ and $\langle \mathcal{Q}' | H_{in}^{(k')} | \mathcal{Q}'' \rangle$ are different from zero. This is a consequence of the fact that all the interaction terms in equation (56) describe different processes.

Assume now with YANG and TIOMNO, that it is possible to attribute types to all the Fermions in such a manner that under time reversal and space reflection we have

$$(57) \quad \hat{H}_a = H_a; \quad \tilde{H}_a = H_a$$

and that further for any given k at least one of the equations

$$(58) \quad \hat{H}_{in}^{(k)} = H_{in}^{(k)}, \quad \tilde{H}_{in}^{(k)} = H_{in}^{(k)}$$

is not satisfied. In our terminology this means that K_0, K'_0 has to be chosen in such a way that (57) holds and (58) does not hold. That such a choice of K_0, K'_0 is possible in more than one way follows from the work of GAMBA.

This choice of types insures that H_a is invariant under time reversal and space reflection but that $H_a + H_{in}^{(k)}$ is not, for any value of k . From this YANG and TIOMNO concluded that the inadmissible interactions are forbidden.

With our definition of the transformation of time reversal and space reflection this conclusion is not correct. For in order to show that for a given k , $H_a + H_{in}^{(k)}$ is not invariant under time and space reflections, we have to show that there exists no pair of functions F and F' such that

$$(59) \quad \exp[iF'](\tilde{H}_a + \tilde{H}_{in}^{(k)}) \exp[-iF'] = H_a + H_{in}^{(k)}$$

and

$$(60) \quad \exp[iF](\hat{H}_a + \hat{H}_{in}^{(k)}) \exp[-iF] = H_a + H_{in}^{(k)}.$$

Because of equation (57) the choice of F and F' is restricted by the conditions

$$(61) \quad \exp[iF']H_a \exp[-iF'] = H_a; \quad \exp[iF]H_a \exp[-iF] = H_a,$$

which expresses the invariance of the admissible interaction Hamiltonian. It follows that, if we want to verify, that to assign types to fermions, is an efficient mechanism to forbid the reactions of the type k , we would have to show that there exist *no* functions F and F' which satisfy

$$(62) \quad \exp[iF']\tilde{H}_{in}^{(k)} \exp[-iF'] = H_{in}^{(k)}; \quad \exp[iF]\hat{H}_{in}^{(k)} \exp[-iF] = H_{in}^{(k)}.$$

In the next section we shall show for the case of the reaction (1), that it is possible to find functions F and F' which satisfy (62), so that in the frame of our definition of the transformations of spacereflection and timereversal the introduction of types is incapable of forbidding reaction (1).

4.3. The existence of an F function which changes the types without affecting the observables. — The reaction (1) is due to the interaction Hamiltonian

$$(63) \quad H_{in}^{(k)} = g \int \sum_{\alpha\beta\gamma\delta} A_{\alpha\beta\gamma\delta} \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\gamma^{e-} \psi_\delta^\nu + \text{Herm. Conj.}$$

where ψ^p , ψ^n , ψ^{e-} , ψ^ν are operators destroying respectively a proton, a neutron, an electron and a neutrino. In the type-philosophy this interaction is excluded as not invariant. In fact, if we take for definiteness $A_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta}$, i.e. the WIGNER and CRITCHFIELD⁽¹⁴⁾ interaction, we have with the types assignment (15) discussed by GAMBA⁽¹³⁾:

$$(64) \quad \tilde{H}_{in}^{(k)} = -H_{in}^{(k)}, \quad \hat{H}_{in}^{(k)} = H_{in}^{(k)}.$$

⁽¹⁴⁾ C. L. CRITCHFIELD: *Phys. Rev.*, **63**, 416 (1943).

⁽¹⁵⁾ The 4 parity types A, B, C, D transform under parity transformations as $\beta, -\beta, i\beta, -i\beta$; the four time reversal types $\alpha, \beta, \gamma, \delta$ transform with $\alpha_1\alpha_2\alpha_3\beta, -\alpha_1\alpha_2\alpha_3\beta, i\alpha_1\alpha_2\alpha_3\beta, -i\alpha_1\alpha_2\alpha_3\beta$ respectively. Then according to reference (13) a possible assignment of types is: $p(A\gamma), n(A\gamma), e^-(C\beta), \nu(C\alpha)$. This assignment excludes all the inadmissible processes in the type philosophy.

If therefore the spinor fields were allowed to transform only in the manner indicated in the footnote the reaction (1) would be forbidden because it is not invariant under space reflections.

We now show, that it is possible to choose F' in such a manner that equation (62) is satisfied. Since all the admissible interactions leave Q and N_h invariant, it follows from equation (61) that F' at least depends on Q and N_h . Q is no longer defined by equation (41) but is now the sum of the charges of all the particles which enter the theory. F' may yet contain other variables, but for the purpose of showing that (62) can be satisfied it is sufficient to study the dependence on N_h . The actual choice of F' is simply

$$(65) \quad F' = \frac{\pi}{2} N_h \quad (F = 0).$$

This gives

$$\exp \left[i \frac{\pi}{2} N_h \right] \tilde{H}_{in}^{(k)} \exp \left[-i \frac{\pi}{2} N_h \right] = - \exp \left[i \frac{\pi}{2} N_h \right] H_{in}^{(k)} \exp \left[i \frac{\pi}{2} N_h \right] = H_{in}^{(k)}$$

the latter because all the matrix element of $H_{in}^{(k)}$ change the nucleon number N_h by ± 2 ⁽¹⁶⁾.

It has thus been shown, that it is not correct to eliminate the interaction term (62) by attributing definite types to all the particles concerned. (62) could of course be excluded by saying that it does not conserve the number of nucleons, but this would be just stating an experimental fact.

A possible way of looking at this result is the following: The observables of a physical system of fields (i.e. one which conserves the number of heavy particles) are all invariant under canonical transformations induced by a function $F(Q, N_h)$. However, the non observable spinors are not insensitive to such a transformation, which may in fact change the type of the heavy particles. But such a change of type cannot have an observable consequence on account of the invariance of the observables. The situation is quite similar to the one discussed in I, when dealing with the Hamiltonian $H = \frac{1}{2} p^2 + (\mathbf{p} \cdot \mathbf{x}) V(|\mathbf{x}|)$. If in this case the criterion of reversibility of a system would be taken to be the invariance of the Hamiltonian under $\mathbf{x} \rightarrow \mathbf{x}$, $\mathbf{p} \rightarrow -\mathbf{p}$ this system would be classed as irreversible. We have shown in I that it is possible to transform this Hamiltonian into $\frac{1}{2} p'^2 - x^2 V(|\mathbf{x}|)$ and this Hamiltonian is invariant under $\mathbf{x} \rightarrow -\mathbf{x}$, $\mathbf{p} \rightarrow -\mathbf{p}$. The equations of motion for the observables remain

⁽¹⁶⁾ This conclusion is true, whatever the assignment of types. For if by a suitable choice of types we prohibit the reaction (1), this implies that either $\tilde{H}_{in}^{(k)}$ or $\tilde{H}_{in}^{\wedge(k)}$ or both are equal to $-H_{in}^{(k)}$. We may accordingly choose either F' or F or both equal to $(\pi/2)N_h$ and thus satisfy equation (62).

the same under this canonical transformation. The point is, that in this system \mathbf{p} is not an observable owing to the fact that all the observables are invariant under $\mathbf{p} \rightarrow \mathbf{p} + \partial F / \partial \mathbf{x}$; fixing the transformation properties of non-observables, we can only decide whether or not a given Hamiltonian is invariant under a certain transformation, but never whether the system it describes is invariant or not. Exactly the same is done by introducing types. We find that a certain Hamiltonian is not invariant under a transformation, but there may be another Hamiltonian leading to the same equations of motion for the observables, which is invariant and we have indeed shown that this is the case.

4.4. *Final remarks.* — To extend the discussion to the other inadmissible processes, we would have to study in more detail the class of functions F and F' selected by equation (61) for the admissible interactions. In the previous section we have only made use of the dependence of F and F' on N_h , but as we have already observed, there may be other occupation numbers not completely eliminated by a discussion of the allowed transitions, so that an F satisfying equation (61) would be of the form $F(Q, N, n, \dots)$, where n is some combination of sums of occupation numbers. The freedom in the choice of the dependence on $n \dots$ could then be used to apply the argument of the previous section to other unwanted processes as for example

$$(66) \quad \mu^+ \rightarrow e^+ + e^+ + e^-.$$

Such a discussion would have to be based on a detailed consideration of the allowed transitions, into which we will not enter here.

What we have seen here is that the consideration of the dependence of F on Q and N_h alone, implies that in speaking of a universal interaction we have not only to exclude from the start those interaction terms which do not conserve the charge, but also those which do not conserve the number of nucleons.

APPENDIX

We want to show that for the system considered in section 2.3 the most general F has the form (40).

The fact that all the interactions (35), (36), (37) are invariant under $K_0 = K_{0y} K_{0m} K_{0a}$ implies that

$$\langle \mathcal{Q}' | \exp[iF] H \exp[-iF] | \mathcal{Q}'' \rangle = \langle \mathcal{Q}' | H | \mathcal{Q}'' \rangle$$

where $\langle \mathcal{Q}' |$ and $| \mathcal{Q}'' \rangle$ are any two states and H is any one of the interactions (35), (36), (37). The left hand side may also be written, with obvious meaning

of the symbols

$$\exp [i(F(\mathcal{Q}') - F(\mathcal{Q}''))] \langle \mathcal{Q}' | H | \mathcal{Q}'' \rangle$$

so that we have, if $\langle \mathcal{Q}' | H | \mathcal{Q}'' \rangle$ does not vanish:

$$(A.1) \quad F(\mathcal{Q}') = F(\mathcal{Q}'') \pmod{2\pi}.$$

Now consider first H to be the electromagnetic interaction of charged mesons. Then, if $|\mathcal{Q}''\rangle$ is a state containing n_m^0 neutral mesons, n_m^+ positively charged mesons and n_m^- negatively charged, n_γ photons and n_p^+ , n_p^- , n_n^+ , n_n^- protons, antiprotons, neutrons and antineutrons $\langle \mathcal{Q}' |$ will be characterized by one of the following sets of occupation numbers in order to give rise to a non vanishing matrix element:

$$\begin{aligned} \langle \mathcal{Q}' |_1 &\equiv n_m^0, n_m^+, n_m^-, n_\gamma \pm (1 \text{ or } 2), n_p^+, n_p^-, n_n^+, n_n^-, \\ \langle \mathcal{Q}' |_2 &\equiv n_m^0, n_m^+ + 1, n_m^- + 1, n_\gamma - 1, n_p^+, n_p^-, n_n^+, n_n^-. \end{aligned}$$

The matrix element of $H_{\gamma m}$ between $\langle \mathcal{Q}' |_1$ and $|\mathcal{Q}''\rangle$ corresponds to a (virtual) process like

$$(A.2) \quad \pi^+ \pm (1 \text{ or } 2) \gamma \rightarrow \pi^{+'}.$$

The matrix element of $H_{\gamma m}$ between $\langle \mathcal{Q}' |_2$ and $|\mathcal{Q}''\rangle$ corresponds to

$$(A.3) \quad \gamma \rightarrow \pi^+ + \pi^-.$$

We have thus, from (A.1), the independent equations

$$\begin{aligned} (A.4) \quad F(n_m^0, n_m^+, n_m^-, n_\gamma + 1, n_p^+, n_p^-, n_n^+, n_n^-) = \\ = F(n_m^0, n_m^+, n_m^-, n_\gamma, n_p^+, n_p^-, n_n^+, n_n^-) \end{aligned}$$

and

$$\begin{aligned} (A.5) \quad F(n_m^0, n_m^+ + 1, n_m^- + 1, n_\gamma - 1, n_p^+, n_p^-, n_n^+, n_n^-) = \\ = F(n_m^0, n_m^+, n_m^-, n_\gamma, n_p^+, n_p^-, n_n^+, n_n^-). \end{aligned}$$

(A.4) is a consequence of the matrix element corresponding to process (A.2), and (A.5) corresponds to process (A.3).

Eq. (A.4) implies that F is independent of the photon number n_γ . Taking account of this fact eq. (A.5) may be rewritten

$$F(n_m^0, n_m^+ + 1, n_m^- + 1, n_p^+, n_p^-, n_n^+, n_n^-) = F(n_m^0, n_m^+, n_m^-, n_p^+, n_p^-, n_n^+, n_n^-).$$

Proceeding by iteration we get

$$F(n_m^0, n_m^+, n_m^-, n_p^+, n_p^-, n_n^+, n_n^-) = F(n_m^0, n_m^+ - n_m^-, 0, n_p^+, n_p^-, n_n^+, n_n^-)$$

or

$$F(n_m^0, n_m^+, n_m^-, n_p^+, n_p^-, n_n^+, n_n^-) = F(n_m^0, 0, n_m^- - n_m^+, n_p^+, n_p^-, n_n^+, n_n^-)$$

according if $n_m^+ \geq n_m^-$; in any case F is just a function

$$F(n_m^0, n_m^+ - n_m^-, n_p^+, n_p^-, n_n^+, n_n^-)$$

if we let $n_m^+ - n_m^-$ to assume positive and negative values.

Proceeding in exactly the same way with the other interactions $H_{m,n}$, $H_{\gamma n}$ we finally get the result (40).

RIASSUNTO

La definizione di «time reversal» data in un precedente lavoro ⁽¹⁾, basata solamente sulle proprietà di trasformazione di osservabili, è estesa al caso delle riflessioni spaziali e sviluppata in modo che nessuna preferenza viene data ai numeri di occupazione dell'impulso, piuttosto che ad altri numeri di occupazione. Questo porta con sé una restrizione considerevole nella classe delle funzioni F con la conseguenza che per un sistema reversibile di campi le osservabili si trasformano in modo completamente definito, mentre le non osservabili non lo fanno. Il formalismo è applicato al problema dei tipi di fermioni; si mostra che l'esclusione delle interazioni che implicano la distruzione di particelle pesanti non può essere giustificata assegnando tipi ai fermioni.

Particelle pesanti instabili in emulsioni nucleari.

M. BALDO, G. BELLIBONI, M. CECCARELLI, M. GRILLI, B. SECHI,
B. VITALE (*) e G. T. ZORN (+)

Istituto di Fisica dell'Università - Padova
Istituto Nazionale di Fisica Nucleare - Sezione di Padova

(ricevuto il 7 Aprile 1955)

Riassunto. — Si presentano i risultati di una analisi sistematica sulle particelle pesanti instabili effettuata mediante pellicole di emulsione nucleare esposte alla radiazione cosmica durante la Spedizione Internazionale di Sardegna del 1953. Dopo una breve introduzione sui metodi sperimentali usati, vengono esposti dati riguardanti i mesoni pesanti, gli iperoni ed i frammenti instabili. Nelle conclusioni del lavoro sono presentate alcune considerazioni sulle frequenze relative dei vari eventi e sulla attendibilità degli schemi di decadimento proposti.

Introduzione.

L'interesse suscitato in questi ultimi anni dalla scoperta dei mesoni pesanti e delle particelle instabili di massa superiore a quella del protone ci ha indotti ad affrontarne lo studio sistematico con la tecnica delle emulsioni nucleari. La presente nota descrive il lavoro compiuto in questo campo dal nostro gruppo durante l'anno passato e perciò contiene anche dati presentati al congresso di Padova dell'Aprile 1954 ⁽¹⁻⁶⁾.

(*) Attualmente presso l'Istituto di Fisica dell'Università, Roma.

(+) Attualmente presso il Brookhaven National Laboratory, Brookhaven, U.S.A.

(¹) G. BELLIBONI, B. SECHI and B. VITALE: *Suppl. Nuovo Cimento*, **12**, 195 (1954).

(²) M. BALDO, G. BELLIBONI, B. SECHI and G. T. ZORN: *Suppl. Nuovo Cimento*, **12**, 220 (1954).

(³) M. BALDO, M. CECCARELLI, M. GRILLI and G. T. ZORN: *Suppl. Nuovo Cimento*, **12**, 257 (1954).

(⁴) M. BALDO, G. BELLIBONI, M. CECCARELLI and B. VITALE: *Suppl. Nuovo Cimento*, **12**, 289 (1954).

(⁵) M. CECCARELLI, M. GRILLI and B. VITALE: *Suppl. Nuovo Cimento*, **12**, 305 (1954).

(⁶) M. GRILLI, B. SECHI and G. T. ZORN: *Suppl. Nuovo Cimento*, **12**, 310 (1954).

All'inizio della nota sono descritti i metodi di osservazione impiegati nella ricerca degli eventi e viene dato qualche cenno sulle tecniche usate nel loro studio. In seguito si trovano un'analisi degli eventi riuniti in opportune categorie e qualche particolare sugli accorgimenti sperimentali applicati di volta in volta. Tutto ciò, pensiamo, nello spirito delle raccomandazioni elaborate dal « Comitato per la normalizzazione delle misure » nel corso del congresso di Padova ⁽⁷⁾.

1. - Parte sperimentale.

1.1. Caratteristiche delle emulsioni nucleari impiegate. - La ricerca qui descritta è stata eseguita con lastre nucleari esposte alla radiazione cosmica mediante palloni liberi nel corso di una spedizione internazionale effettuata in Sardegna nell'estate del 1953 ⁽⁸⁾.

Sono state impiegate emulsioni Ilford G5 senza supporto (pellicole) dello spessore di 600 μ ed aventi dimensioni di 10 \times 15 cm. Queste pellicole erano riunite in blocchi di 40 unità i quali, durante il volo, avevano il lato maggiore posto verticalmente. Ogni blocco era contenuto in un involucro di alluminio a tenuta di vuoto e risultava circondato da una quantità di sostanza solida pari a circa 3 g/cm². Al termine dell'esposizione le pellicole, fatte aderire a lastre di vetro, furono sviluppate a Bristol con il procedimento consueto; ⁽⁹⁾ le lastre furono quindi tagliate valendosi di segni di riferimento ottenuti in precedenza irradiando i blocchi con sottili pennelli di raggi X. I bordi del vetro venivano così a costituire un riferimento per stabilire la corrispondenza geometrica tra le varie lastre.

Abbiamo trovato in seguito conveniente, per ovviare ad imprecisioni nel taglio delle lastre, di fissare queste su dei rettangoli di plexiglas, praticamente identici, i cui bordi costituivano un riferimento più preciso. Abbiamo potuto così ridurre a meno di 50 μ gli spostamenti trasversali fra le porzioni di una stessa traccia osservata in pellicole successive, rendendo possibile l'inseguimento immediato di tracce aventi densità di grani superiore a due volte il valore minimo. Per ritrovare invece in lastre successive le tracce prossime al minimo è sempre stato conveniente valersi di riferimenti locali, cioè di tracce piuttosto scure, parallele e vicine a quella da seguire.

Nella tabella I sono riportate le caratteristiche più significative dei due blocchi di lastre impiegati.

(7) *Proposte di normalizzazione: Suppl. Nuovo Cimento*, **12**, 474 (1954).

(8) J. DAVIES and C. FRANZINETTI: *Suppl. Nuovo Cimento*, **12**, 480 (1954).

(9) C. F. POWELL: *Phil. Mag.*, **44**, 219 (1953).

TABELLA I.

| Contrassegno del blocco | Altezza media del volo | Durata del volo | Densità di grani/50 μ al plateau |
|----------------------------|---------------------------|--------------------|---|
| S_6 | 23,6 km | 6h 50' | 10,2 (*) |
| | | | 9,7 |
| | | | 9,4 |
| S_{27} | 27,0 km | 7h | 13,7 |
| | | | 12,0 |
| | | | 10,7 |

(*) I tre numeri indicano il valor medio e gli estremi inferiore e superiore.

1'2. *Ricerca ed identificazione degli eventi.* — Le emulsioni nucleari sono state osservate al microscopio in parte con un ingrandimento di 208 diametri ottenuto dalla combinazione di un obbiettivo a secco $26\times$ con oculari $8\times$ e in parte con un ingrandimento di 240 diametri ottenuto con l'obbiettivo « Koritzka $30\times$ » ad immersione. Non possiamo però indicare con sicurezza quale dei due sistemi ottici sia il più conveniente agli effetti della rapidità di ritrovamento degli eventi desiderati.

Nel corso dell'osservazione, pur senza variare di regola il sistema ottico, sono stati esaminati con particolare attenzione gli estremi di tutte le tracce dovute a particelle di massa vicina alla protonica, alla ricerca di eventuali tracce sottili ad esse collegate. Molti degli eventi qualificati dagli osservatori come stelle di σ , eventi $\pi \rightarrow \mu$, stelle « piccole » ($N_h + n_s \leq 5$) e stelle doppie sono stati poi riesaminati ed identificati con sicurezza. Il tempo complessivo di osservazione è stato di circa 9000 ore; non è altrettanto facile invece valutare il tempo impiegato per la revisione.

La revisione delle stelle piccole è stata quella che ha portato all'identificazione del maggior numero di particelle pesanti instabili sfuggite come tali all'osservazione ordinaria. Nel corso di essa abbiamo cercato di stabilire il verso del moto delle particelle facenti parte di una stella in modo da individuarne il primario. Sono stati trovati così parecchi eventi nei quali l'energia totale messa in gioco era notevolmente superiore all'energia cinetica della particella primaria; questi eventi avevano quindi richiesto anche una cessione dell'energia a riposo del primario che doveva allora essere considerato instabile.

Oltre all'osservazione ordinaria è stato usato un altro metodo di ricerca consistente nel seguire a ritroso tracce di mesoni π (eventi σ e $\pi \rightarrow \mu$) fino alla loro origine, che risultava talvolta essere il punto di decadimento di una particella pesante instabile ⁽¹⁰⁾. Alcuni dei risultati relativi a questo metodo

⁽¹⁰⁾ D. LAL, Y. PAL and B. PETERS: *Proc. Ind. Ac. Sci.*, **38**, 277 (1953).

sono raccolti in tabella II.

TABELLA II.

| | π^+ | π^- | $\pi^+ + \pi^-$ |
|---|---------|---------|-----------------|
| Numero di tracce seguite | 238 | 292 | 530 |
| Mesoni aventi origine nell'emulsione. | 128 | 170 | 298 |
| Mesoni provenienti dall'esterno | 37 | 49 | 86 |
| Tracce perdute durante l'inseguimento | 70 | 68 | 138 |
| Lunghezza di traccia seguita (cm) | 481 | 401 | 882 |

L'insieme dei risultati ottenuti con i diversi metodi di ricerca e di revisione è raccolto in tab. III. Dai dati di questa tabella non è possibile ricavare direttamente le frequenze assolute e relative dei vari tipi di eventi, che cercheremo invece di valutare al termine del lavoro esaminando i probabili fattori di perdita e mettendo i nostri dati in relazione con quelli di altri Autori.

TABELLA III.

| Metodo di ritrovamento | Vol. osser. (cm ³) | Stelle | Eventi σ | Eventi $\pi^- \rightarrow \mu$ | K_L | τ | τ in volo | K_{π^3} | K^- | Y^\pm | Y^- | Λ^0 | F^* |
|---------------------------------------|--------------------------------|--------|-------------------------|--------------------------------|-------|--------|----------------|-------------|-------|---------|-------|-------------|---------|
| OSSERVAZIONE ORDINARIA | 70 | 27 000 | 1 750 | 1 340 | 12 | 3 | | | 3 | 1 | | | 3 + (2) |
| INSEGUIMENTO A RITROSO DEI π | | | n. delle tracce seguite | | | | | | | | | | |
| | | | π^+ 238 | π^- 292 | | | | | | | | | |
| REVISIONE | | | n. eventi revisionati | | | | | | | | | | |
| Eventi σ | | | 1 200 | | | | | | | | | | |
| Stelle a due rami | | | 2 500 | | | | | | | | | | |
| Stelle piccole ($N_h + n_s \leq 5$) | | | 12 000 | | | | | | | | | | |
| Stelle doppie | | | 96 | | | | | | | | | | |
| Totale | | | | | | | | | | | | | |

1'3. *Metodi di misura.* — Tutti gli eventi descritti in questo lavoro sono stati analizzati con i metodi e le tecniche di misura usuali, seguendo di regola le raccomandazioni di unificazione già citate (7).

Le misure di « scattering » sono state effettuate su microscopi « Koritzka » con sospensione a lame, per i quali il « noise » è assai piccolo. Il valore del « noise » totale nella misura di tracce sottili e grigie è risultato di circa $0,1 \mu$ e comunque mai superiore a $0,2 \mu$.

La massa delle particelle arrestantisi nell'emulsione è stata determinata con il metodo dello « scattering a sagitta costante » (11) per le tracce di lunghezza inferiore ad un centimetro; la porzione di traccia eventualmente eccedente è stata invece sottoposta a misure di scattering con celle costanti. La sequenza di celle per lo scattering a sagitta costante era tale da fornire una sagitta media di 1μ per particelle di 1000 masse elettroniche. Essa è stata tarata con le tracce di 20 protoni accelerati artificialmente, la cui massa media è risultata di $1870 \pm 70 m_e$. È stata anche determinata la leggera variazione della sagitta in funzione del percorso per poterne poi tenere conto nel calcolo delle masse. Il contributo della distorsione della gelatina, distorsione misurata in prossimità della traccia in studio, è stato eliminato di regola dai singoli valori delle sagitte e solo eccezionalmente impiegando il metodo delle differenze di ordine superiore (12).

Le misure di scattering di tracce sottili e grigie sono state fatte con celle costanti e sagitta media di circa 4 volte il valore del « noise ». Nei casi in cui la particella attraversava più pellicole le misure sono state fatte solamente in quelle meno distorte ed anche in questo caso il contributo della distorsione è stato eliminato dalle singole sagitte. I valori di $p\beta$ sono stati ottenuti, da quelli della sagitta media, impiegando le costanti di scattering riportate da VOJVODIC e PICKUP (13), con « cut-off » a 4 volte il valor medio.

La massa delle particelle che si arrestano in emulsione è stata anche determinata, oltre che dallo scattering, dal valore della ionizzazione specifica in funzione del percorso. Pensiamo sia opportuno illustrare brevemente il metodo da noi seguito perchè lo riteniamo tra i più idonei alla misura delle masse nella tecnica delle emulsioni senza supporto.

La traccia da misurare veniva seguita a ritroso e sul tratto di essa avente densità di grani compresa tra 60 e 40 grani/50 μ veniva effettuato, con criteri uniformi, il conteggio dei grani ed insieme si determinavano i percorsi residui corrispondenti. I conteggi, eseguiti soltanto nello strato centrale delle singole

(11) S. BISWAS, E. C. GEORGE and B. PETERS: *Proc. Ind. Ac. Sci.*, **38**, 418 (1953); C. DILWORTH, S. J. GOLDSACK and L. HIRSCHBERG: *Nuovo Cimento*, **11**, 113 (1954).

(12) S. BISWAS, E. C. GEORGE, B. PETERS and M. S. SWAMY: *Suppl. Nuovo Cimento*, **12**, 369 (1954).

(13) L. VOJVODIC and E. PICKUP: *Phys. Rev.*, **85**, 91 (1952).

pellicole per eliminare l'effetto dei gradienti di sviluppo, si estendevano ad un totale di almeno 1000 grani. Quindi venivano cercate, nelle vicinanze dei punti in cui la traccia era stata misurata, delle tracce di protoni il più possibile simili ad essa per densità e pendenza. Anche queste tracce erano sottoposte ad analoghe misure di densità e seguite fino al loro punto d'arresto per determinarne il percorso. Impiegando la relazione, di facile verifica, secondo cui il rapporto tra i percorsi residui di due particelle ugualmente ionizzanti è uguale a quello tra le loro masse, si otteneva immediatamente la massa incognita di una particella.

Per stabilire la massa delle particelle veloci che non si arrestavano in emulsione abbiamo impiegato il metodo di Fowler⁽¹⁴⁾ valendoci esclusivamente di curve $g^* - \bar{\alpha}_{100\mu}$ da noi determinate. Poichè questo metodo è stato applicato soprattutto nello studio dei secondari dei mesoni K , ne rimandiamo la descrizione più particolareggiata al paragrafo in cui si tratterà di queste particelle.

Le misure di scattering e di percorso sono state condotte valendosi di oculari micrometrici le cui indicazioni erano state tarate in precedenza con un comparatore « Zeiss » assai preciso. La determinazione dell'energia dai valori del percorso è stata fatta con la relazione indicata da G. BARONI e coll.⁽¹⁵⁾ supponendo che le emulsioni all'atto dell'esposizione fossero secche e dello spessore di 600 μ .

I valori delle masse delle particelle elementari da noi usati nel corso di questo lavoro sono i seguenti:

$$m_p = 1836,13 m_e, \quad m_n = 1838,7 m_e, \quad m_{\pi^+} = 273,3 m_e,$$

$$m_{\pi^-} = 272,7 m_e \quad \text{e} \quad m_{\pi^0} = 263,9 m_e.$$

Gli errori attribuiti alle singole misure sono « standard deviations » calcolate secondo i suggerimenti del Comitato per la normalizzazione delle misure. I valori medi di una serie di più misure sono stati calcolati pesando queste inversamente ai quadrati dei loro errori relativi.

2. — Descrizione degli eventi.

— MESONI PESANTI.

In questa parte sono descritti gli eventi dovuti a particelle instabili di circa 1000 m_e (mesoni K) elencati nella tab. III e i risultati delle relative misure.

⁽¹⁴⁾ P. H. FOWLER: *Phil. Mag.*, **41**, 169 (1950).

⁽¹⁵⁾ G. BARONI, C. CASTAGNOLI, G. CORTINI, C. FRANZINETTI e A. MANFREDINI: Comunicazione BS 9, C. E. R. N., Luglio 1954.

Per la classificazione dei casi di decadimento spontaneo in cui sia emessa una sola particella carica, impiegheremo il simbolo K_{an} , proposto da B. ROSSI⁽¹⁶⁾, dove i due indici corrispondono rispettivamente al simbolo del secondario visibile ed al numero dei prodotti di decadimento. Impiegheremo inoltre il simbolo K_l per indicare genericamente i mesoni K che presentano un solo secondario carico, debolmente ionizzante.

Abbiamo infine impiegato simboli costituiti da minuscole greche soltanto per indicare i mesoni dei quali sia completamente noto lo schema di decadimento.

1) *Mesoni K_l* - Nel corso di questa ricerca abbiamo osservato 12 eventi in cui una particella di circa 1000 masse elettroniche si arresta in emulsione ed emette un secondario prossimo al minimo di ionizzazione. I risultati dello studio di questi mesoni K_l sono contenuti in tab. IV.

Quattro di questi eventi, indicati nella tabella con i simboli $K_l\text{-Pd}_3$, $K_l\text{-Pd}_7$, $K_l\text{-Pd}_8$ e $K_l\text{-Pd}_{11}$, presentano un particolare interesse perchè di essi abbiamo potuto determinare le masse delle particelle secondarie. Le quattro tracce secondarie erano infatti assai lunghe e poco inclinate rispetto al piano dell'emulsione.

Per effettuare le misure di massa abbiamo costruito le curve della ionizzazione specifica in funzione di $p\beta$ usando tracce di mesoni π di energia prossima a quella dei secondari dei K_l , lunghe almeno 6 mm per lastra. La traccia del secondario e quelle impiegate per la taratura dovevano inoltre attraversare le medesime pellicole in punti distanti non più di qualche centimetro.

Le misure di ionizzazione e scattering della traccia del secondario e di quelle dei pioni di taratura venivano effettuate in ogni singola lastra. Abbiamo però visto in

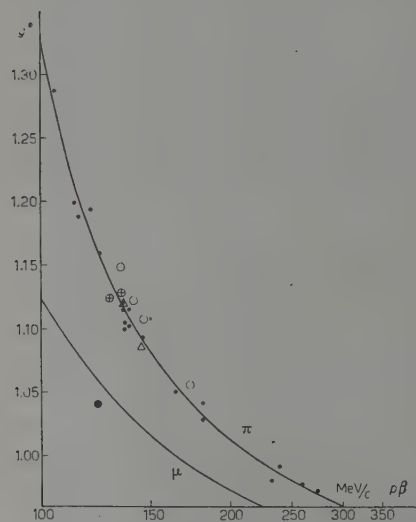


Fig. 1. - Valori della densità di grani normalizzata, g^* , in funzione di $p\beta$ per le tracce di 20 mesoni π e per quelle dei secondari di 4 mesoni K_l :
 • mesoni π ; \otimes secondario del $K_l\text{-Pd}_3$; \triangle secondario del $K_l\text{-Pd}_7$;
 ● secondario del $K_l\text{-Pd}_8$; ○ secondario del $K_l\text{-Pd}_{11}$.

seguito che i punti $g^*\text{-}p\beta$ relativi alle diverse lastre potevano essere riuniti con buona approssimazione da un'unica curva.

In fig. 1 sono riportati i punti rappresentativi delle tracce secondarie e di

(16) B. ROSSI: *Rendiconti Scuola di Varenna* 1954 (in corso di pubblicazione sul *Nuovo Cimento*).

TABELLA IV.

| | | | P R I M A R I O | | | S E C O N D A R I O | | | | | | |
|-------------------------|--------------------------|----------------|----------------------------|---------------------------|----------------------|---------------------|-------------------|----------------------|-------------------------|--|--------------------|--------------|
| Evento | Stella d'ori- gine | θ_p (1) | Percorso in (mm) (2) | N. lastre attraversate | Massa | | | Percorso (mm) (2) | Lungh. per lastra | $p\beta$ alla emis- sione (MeV/c) | Massa (m_e) | Osservatore |
| | | | | | ($\bar{\alpha}$ -R) | Sagitta costante | Cella costante | | | | | |
| K_I -Pd ₃ | 21+7n | — | 42,5 (t) | 10 | — | — | 964±60 | 21 (o) | 6,5 | 160±9 | 273±33 | G. GESUATO |
| K_I -Pd ₄ | 16+1p | 39° | 19,4 (t) | 2 | 922±195 | 965±129 | 975±200 | 40 (o) | 1,85 | 110±15 | — | G. GESUATO |
| K_I -Pd ₅ | 5+1p | 121° | 19,3 (t) | 11 | 1001±360 | 845±336 | — | 3,7 (o) | 1,8 | 70±16 | — | G. GESUATO |
| K_I -Pd ₆ | 14+8p | 178° | 17,2 (t) | 18 | 865±292 | — | 850±100 | 3,3 (o) | 0,72 | — | — | G. GESUATO |
| K_I -Pd ₇ | 6+0p | 66° | 36,5 (t) | 5 | 957±176 | 947±99 | 943±60 | 24 (o) | 17 | 159±10 | 272±30 | D. NARCISO |
| K_I -Pd ₈ | 16+8z | 88° | 26,9 (t) | 16 | 895±225 | — | 1180±140 | 44,4 (o) | 16,4 | 149±8 | 194±25 | D. NARCISO |
| K_I -Pd ₉ | 19+1p | 85° | 12,0 (t) | 5 | 1420±360 | — | 1060±100 | 21 (o) | 3,3 | 153±36 | — | G. BIANCHINI |
| K_I -Pd ₁₀ | 11+4p | 38° | 25,2 (t) | 14 | — | — | 1100±160 | 10 (o) | 1 | — | — | A. BERNARDI |
| K_I -Pd ₁₁ | 36+4n | — | 4,9 (t) | 3 | 1056±225 | — | 1150±110 | 44,8 (o) | 10 | 172±9 | 286±28 | B. SECCHI |
| K_I -Pd ₁₂ | 20+23z | 133° | 5,4 (t) | 2 | 964±300 | — | 964±300 | 4,2 (o) | 0,67 | — | — | D. NARCISO |
| K_I -Pd ₁₃ | 21+5n | — | 0,4 (t) | 1 | — | — | — | 20 (o) | 0,92 | — | — | D. NARCISO |
| K_I -Pd ₁₄ | 13+9p | 96° | 44,7 (t) | 7 | 1120±200 | 870±85 | 955±60 | 25,43 (o) | 2,2 | — | — | O. GOMIERO |

⁽¹⁾ In tutte le tabelle indichiamo con θ_p l'angolo nello spazio tra la traccia della particella instabile e quella del primario della stella d'origine.

I simboli impiegati sono di regola quelli riportati nel *Suppl. Nuovo Cimento*, 12, 171-72 e 459-60 (1954).

⁽²⁾ In tutte le tabelle diamo il « percorso totale » (t) per le particelle che si arrestano in emulsione e il « percorso osservato » (o) per quelle che escono dal blocco o che disintegrano in volo.

quelle di taratura, insieme alla curva che meglio si adatta a questi ultimi, mentre in fig. 2 è riportato il corrispondente spettro di masse. Da esso risulta che assai probabilmente i secondari dei mesoni K_L -Pd₃, K_L -Pd₇ e K_L -Pd₁₁ sono dei mesoni π , mentre quello del K_L -Pd₈ è un mesone μ .

I tre secondari identificati come mesoni π sono di energia assai simile ($p\beta$ di 160 ± 9 , 159 ± 10 e 172 ± 9 MeV/c) (*), fatto che sembra confermare il

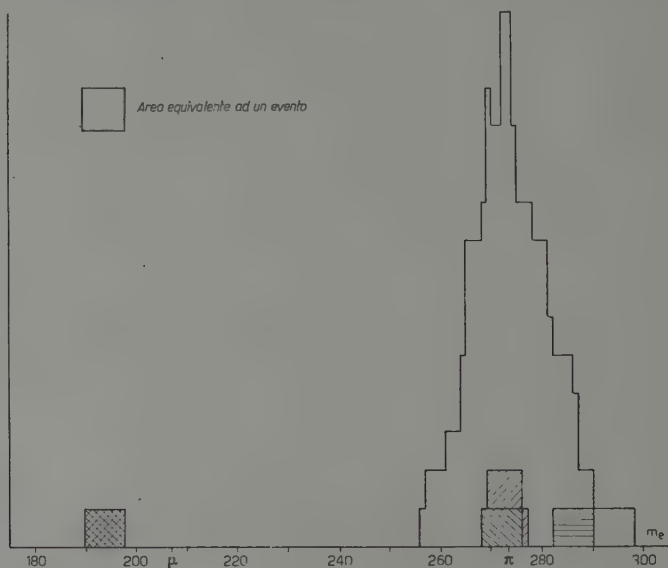


Fig. 2. - Spettro delle masse di 20 mesoni π e dei secondari di 4 mesoni K_L . \square mesoni π ; ▨ secondario del K_L -Pd₃; ▩ secondario del K_L -Pd₇; \blacksquare secondario del K_L -Pd₈; \blacksquare secondario del K_L -Pd₁₁.

decadimento in due corpi già proposto in passato dal gruppo di Bristol (mesoni γ) (¹⁷). Attribuendo alla particella primaria una massa di $962 \pm 38 m_e$, media delle masse dei tre mesoni K_L -Pd₃, K_L -Pd₇ e K_L -Pd₁₁, ed al pione secondario un'energia di 105 ± 4 MeV, si ottiene per la particella neutra una massa di $296 \pm 60 m_e$ e sembra perciò ragionevole proporre lo schema di decadimento:

$$K_{\pi 2}^{\pm} \rightarrow \pi^{\pm} + \pi^0 + Q.$$

I valori della massa del mesone $K_{\pi 2}$ e dell'energia Q che meglio si adattano allora ai nostri dati sperimentali, sono rispettivamente di $953 \pm 30 m_e$ e 213 ± 15 MeV.

(*) Nel calcolo degli errori non abbiamo tenuto conto dell'errore relativo alla costante di scattering. Questo fatto d'altra parte non modifica le nostre conclusioni sulla massa dei secondari e sulla relazione tra le loro energie.

(¹⁷) M. G. K. MENON e C. O'CEALLAIGH: *Proc. Roy. Soc.*, **221 A**, 292 (1954).

TABELLA V.

| Evento | PRIMARIO | | | | SECONDARIO ($p_a \geq p_b \geq p_c$) | | | | Q (MeV) | Osservatore |
|-------------------------|------------------|---------------------|---------------|--------|--|------|-----------------------|------------------|--------------|-------------------------|
| | Percorso (mm) | Stella d'origine | ϑ_p | Carica | Tempo di volo (s) | Ramo | Percorso (mm) | Energia (MeV) | Identità | Angoli tra le tracce |
| τ -Pd ₃ | 6,7 (t) | 21+13p | 17° | + | $0,9 \cdot 10^{-10}$ | a | $125,2 \pm 0,6$ (t) | $41,3 \pm 0,5$ | π^+ | 72° |
| | | | | | | b | $12,9 \pm 0,3$ (t) | $27,6 \pm 0,4$ | π^+ | 136° |
| | | | | | | c | $1,3 \pm 0,01$ (t) | $7,3 \pm 0,1$ | π^- | 158° |
| τ -Pd ₄ | 7,4 (t) | 10+1p | 40° | ? | $1,0 \cdot 10^{-10}$ | a | 9,0 (o) | $(33,9 \pm 0,3)$ | — | 100° |
| | | | | | | b | $12,3 \pm 0,5$ (t) | $26,9 \pm 1,3$ | π^- | 120° |
| | | | | | | c | $4,7 \pm 0,2$ (t) | $15,2 \pm 0,3$ | π^+ | 140° |
| τ -Pd ₅ | 12,1 (t) | 14+4n | | ? | $1,3 \cdot 10^{-10}$ | a | 9,0 (o) | $(33,5 \pm 1,5)$ | — | 100° |
| | | | | | | b | $9,2 \pm 0,5$ (t) | $22,5 \pm 0,8$ | π^- | 128° |
| | | | | | | c | $7,2 \pm 0,4$ (t) | $19,6 \pm 0,7$ | π^+ | 132° |
| τ -Pd ₆ | 9,5 (t) | 12+3n | | + | $1,1 \cdot 10^{-10}$ | a | $26,4 \pm 0,8$ (t) | $42,2 \pm 0,4$ | π^- | 27° |
| | | | | | | b | $16,0 \pm 0,5$ (t) | $31,2 \pm 0,4$ | π^+ | 157° |
| | | | | | | c | $0,093 \pm 0,005$ (t) | $1,6 \pm 0,1$ | π^+ | 96° |
| τ -Pd ₇ | 10,2 (t) | 4+2p | 5° | ? | $1,1 \cdot 10^{-10}$ | a | $21,6 \pm 0,5$ (t) | $37,3 \pm 0,5$ | π^+ | 93° |
| | | | | | | b | 16,0 (o) | $(33,3 \pm 0,5)$ | — | 111° |
| | | | | | | c | $0,71 \pm 0,02$ (t) | $5,1 \pm 0,3$ | π^- | 156° |
| τ -Pd ₈ | 14,2 (t) | 16+0p | 80° | + | $1,5 \cdot 10^{10}$ | a | $12,6 \pm 0,3$ (t) | $27,1 \pm 0,4$ | π^+ | 111° |
| | | | | | | b | $10,9 \pm 0,6$ (t) | $24,9 \pm 0,9$ | π^+ | 119° |
| | | | | | | c | $8,3 \pm 0,5$ (t) | $21,3 \pm 0,8$ | π^- | 130° |

Riteniamo di scarso interesse discutere singolarmente i dati relativi agli altri mesoni K_L e per essi rimandiamo alla corrispondente tabella.

2) *Mesoni τ .* - a) *Decadimenti a riposo.* - Nel corso di questa esperienza sono stati identificati 6 casi di decadimento a riposo di mesoni τ in tre pioni carichi emessi in un piano con un'energia cinetica complessiva di circa 75 MeV. I dati sperimentali ad essi relativi sono raccolti in tab. V.

Tre di questi eventi sono « completi », cioè hanno tutti i pioni secondari che terminano in emulsione; per essi quindi è sicura la determinazione del segno della carica, positivo in tutti e tre i casi. Gli altri tre eventi hanno due soli secondari che si arrestano nel blocco, ed essendo questi, in tutti e tre i casi di segno opposto, il segno del primario rimane indeterminato. Le energie cinetiche dei secondari che si arrestano in emulsione sono state ottenute dalla misura del loro percorso totale. Per i secondari che escono dal blocco o che hanno subito interazioni con possibile perdita di energia, l'energia cinetica è stata invece ricavata da quella dei secondari che si arrestano in emulsione e dagli angoli tra le tracce, nell'ipotesi che i tre rami siano tutti dovuti a pioni e che debba annullarsi la quantità di moto totale.

Le tracce dei pioni sono complanari entro circa un grado in tutti i decadimenti osservati, eccettuato l'evento τ -Pd₇, in cui una delle tracce forma un angolo di circa 4° con il piano delle altre. Questo scostamento dalla complanarità è però probabilmente solo un effetto della distorsione.

Il valor medio di Q per gli eventi qui riportati risulta di $75,4 \pm 0,4$ MeV, da cui deriva per il mesone τ una massa di 967,3 m_e qualora si attribuisca ai tre pioni una massa totale di 819,3 m_e .

b) *Un esempio di decadimento in volo.* - Durante la revisione delle stelle piccole è stato osservato un evento che può essere interpretato come il decadimento in volo di un mesone τ (Tav. 1). I dati ad esso relativi sono raccolti in tab. VI.

L'evento appare come una stella a 4 rami, uno dei quali, molto lento, proviene da una disintegrazione nucleare. Gli altri tre rami sono piuttosto veloci e non complanari. Due di essi si arrestano in emulsione e risultano dovuti a pioni; il terzo, pure dovuto ad un mesone leggero, esce invece dal blocco.

Supponendo che l'evento sia dovuto al decadimento in volo di un τ , la risultante delle quantità di moto dei tre secondari dovrà essere uguale alla quantità di moto del primario e quindi avere componente nulla nel piano normale alla sua direzione di moto. È possibile perciò determinare, dalla quantità di moto dei due pioni che si arrestano, quelle del terzo secondario e del primario e le corrispondenti energie. Nel sistema del laboratorio l'energia del terzo secondario risulta di 37,6 MeV e quella del mesone τ , all'atto del decadimento, di 3,8 MeV. Questi valori sono del tutto compatibili con l'angolo medio di scattering e con la densità di grani delle rispettive tracce.

200 μ

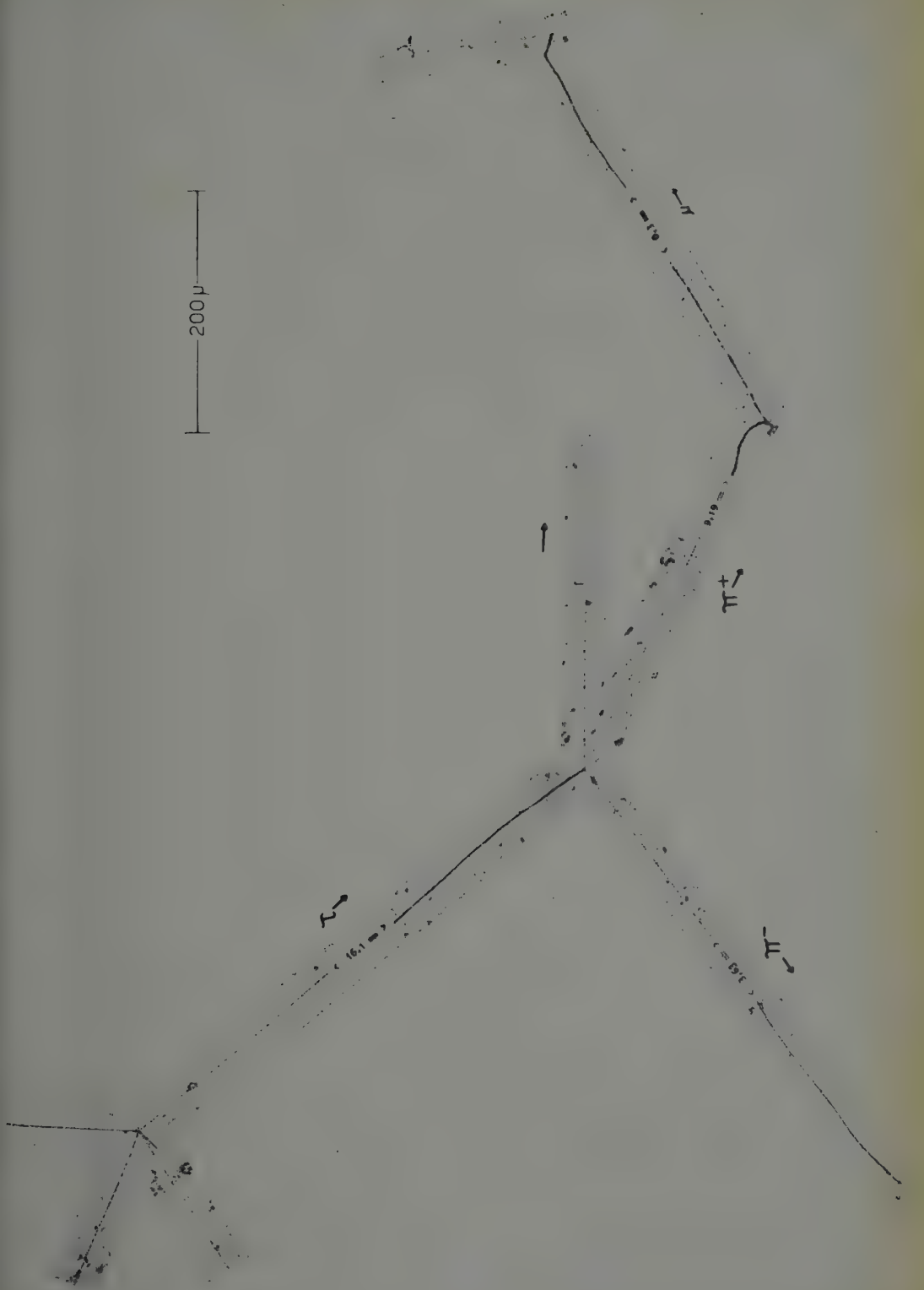


TABELLA VI.

| | P R I M A R I O | S E C O N D A R I | | |
|---|---------------------------------------|--|-----------|------------------------------|
| | | <i>a</i> | <i>b</i> | <i>c</i> |
| Stella d'origine | 5+0n | | | |
| Percorso (mm) | 16,5 (o) Perc. residuo = 170 μ | 9,49 \pm $\pm 0,15$ (t) | 14 (o) | 4,02 \pm $\pm 0,10$ (t) |
| Identità | K | π^+ ($\pi \rightarrow \mu \rightarrow e$) | (π) | π^- (σ_0) |
| <i>p</i> (MeV/c) (S.L.) | 61,6 (*) | 83,3 | (109,0) | 64,1 |
| Energia cinetica (MeV) (S.L.) | 3,8 (*) | 23,0 | (37,6) | 14,0 |
| Angolo di inclinazione (Dip) | 52° aria | 82° aria | 60° vetro | 34° vetro |
| Angolo con il primario nel piano dell'emulsione | — | 135° | 124° | 270° |
| Angolo nello spazio con il primario | — | 44° | 149° | 116° |
| Tempo di volo (s) | 1,6 · 10 ⁻¹⁰ | | | |

(*) All'atto del decadimento.

Inoltre la risultante degli impulsi dei due secondari che si arrestano in emulsione è complanare, entro 3 gradi, con le tracce del terzo secondario e del primario, come da prevedersi nell'ipotesi di un decadimento in volo.

Il *Q* della reazione, differenza tra l'energia cinetica complessiva dei secondari e quella residua del primario, è di $70,8 \pm 3,0$ MeV, in soddisfacente accordo con quello del decadimento a riposo.

La coerenza interna dell'evento che risulta quando esso sia interpretato come decadimento in volo ci fa ritenere poco attendibile una diversa interpretazione; quella, ad esempio, che si tratti del decadimento non complanare di un mesone τ negativo catturato da un nucleo.

c) Un evento K_{π^+} . - Dal punto d'arresto di un mesone K ha origine la traccia di un mesone π^+ che si arresta in emulsione dopo un percorso di 0,90 mm e dà luogo al caratteristico decadimento $\mu \rightarrow e$. L'insieme delle misure effettuate sulle varie tracce ci permette di escludere che l'evento sia dovuto al decadimento diretto di un mesone K in un μ .

A meno di voler introdurre un nuovo tipo di particella, si può considerare questo evento come dovuto al decadimento spontaneo di un mesone τ in un

solo pione carico ed una o più particelle neutre, decadimento in possibile concorrenza con quello ordinario ⁽¹⁸⁾.

L'evento potrebbe essere anche dovuto alla cattura nucleare di un mesone K negativo e successiva emissione di un mesone π^+ . Questa ipotesi ci sembra però meno attendibile della precedente, perchè implicherebbe un meccanismo di assorbimento piuttosto complesso, difficilmente conciliabile con l'assenza di altre particelle ionizzanti, oltre il mesone π^+ , provenienti dal punto d'arresto del mesone K.

I dati sperimentali riguardanti questo evento sono raccolti in tab. VII.

TABELLA VII.

| | Primario | Secondario |
|-----------------------------|----------------------|--|
| Stella d'origine | $8 + 3n$ | — |
| Percorso (mm). | 13,2 (t) | 0,90 (t) |
| Energia (MeV). | 47,0 | 5,9 |
| Massa (m_0) | 966 ± 100 | 220 ± 70 |
| Identità. | K | π^+ ($\pi \rightarrow \mu \rightarrow e$) |
| Tempo di volo (s) | $1,4 \cdot 10^{-10}$ | — |

3) *Mesoni K negativi*. — Nel corso di questa esperienza sono stati trovati sei eventi in cui una particella di carica unitaria e di circa 1000 m_e dà luogo ad una disintegrazione nucleare nel suo punto di arresto in emulsione. Tutte le particelle hanno velocità sensibilmente nulla nel punto in cui avviene la disintegrazione. I valori ottenuti per le loro masse sono tali da far senz'altro escludere che si tratti di ordinari mesoni σ ed anche, per quasi tutti gli eventi, che si tratti di frammenti instabili.

Tre di questi eventi sono stati identificati nel corso dell'osservazione normale, altri due durante la revisione delle stelle piccole ed uno infine durante la revisione sistematica di 1200 mesoni σ .

Questa revisione è stata fatta determinando la massa dei mesoni dalla

⁽¹⁸⁾ R. E. MARSHAK: *Meson Physics*, pag. 347; A. PAIS: *Phys. Rev.*, **86**, 663 (1953); R. DALITZ: *Proc. Phys. Soc.*, **66 A**, 710 (1953).

densità dei grani delle loro tracce per un tratto di 250μ preso a 2 mm dal punto di arresto. In fig. 3 è riportato lo spettro di massa di una parte di questi mesoni e di alcuni protoni di confronto (³).

I dati relativi ai sei mesoni K^- ed alle loro stelle di cattura sono raccolti nella tab. VIII. Nel primo di questi eventi la disintegrazione nucleare dovuta all'assorbimento del K^-

è accompagnata dall'emissione di un probabile mesone π^- che si arresta senza visibile interazione o decadimento. Nei tre eventi successivi le stelle di cattura

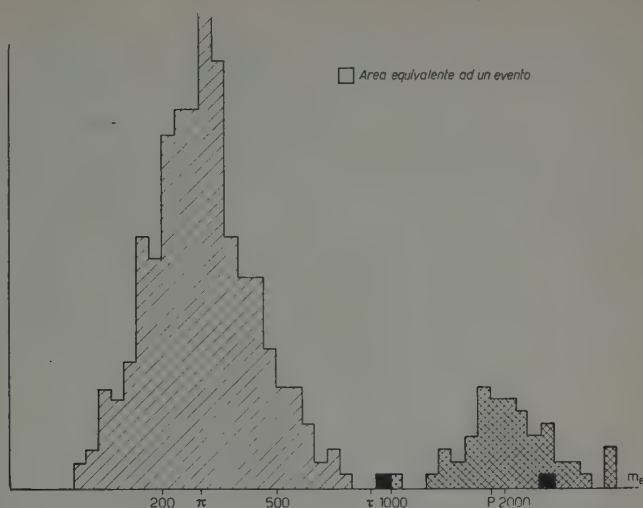


Fig. 3. - Spettro della massa di 320 mesoni π (▨), di 56 protoni (■) e di due particelle pesanti instabili negative (■).

TABELLA VIII.

| | PRIMARIO | | | STELLA DI CATTURA | | | | | Osservatore |
|---------------------------------|--------------------------------------|---------------------|------------------|--|-------------|----------------------|---------|------------------------------|---------------|
| Evento | Massa (m_0) | Stella d'origine | Percorso (mm) | Identità ed energia (MeV) dei rami ionizzanti della stella | | | | Energia visibile (MeV) | |
| K ⁻ -Pd ₁ | 1070 ± 160 | 6 + 1p | 11,2 (t) | p 42 | p 17 | π ⁻ 37 | | 98 + $m_\pi \cdot c^2$ | D. SOTTOVIA |
| K ⁻ -Pd ₂ | 985 ± 150 | 9 + 1p | 47,0 (t) | p 13 | p 11 | | | 24 | A. BERNARDI |
| K ⁻ -Pd ₃ | 987 ± 120 | 5 + 1n | 75,4 (t) | p 7 | α 7 | p 90 | p 45 | 149 | G. CALTABIANO |
| K ⁻ -Pd ₄ | 870 ± 300 | Fuori blocco | 20,2 (t) | p 12 | p 17 | p 22 | | 51 | D. SOTTOVIA |
| K ⁻ -Pd ₅ | 1130 ⁺⁵⁰⁰ ₋₃₀₀ | Fuori blocco | 11,9 (t) | p 65 | (π) ~ 35 | | | 100 + $m_\pi \cdot c^2$ | A. BERNARDI |
| K ⁻ -Pd ₆ | 1190 ⁺⁴⁰⁰ ₋₃₀₀ | 5 + 3p | 1,1 (t) | p 10 | (π) ~ 50 | | | 60 + $m_\pi \cdot c^2$ | V. CHIARATTI |

mostrano l'emissione di soli protoni o particelle α . Negli ultimi due, infine, la cattura è accompagnata dall'emissione di una traccia sottile, in ambedue i casi molto inclinata, e che con discreta attendibilità può essere identificata con un mesone leggero. Nessuno dei rami uscenti dalle stelle di cattura sembra essere dovuto a un iperone o ad un frammento instabile.

Le tracce dei sei mesoni K^- , nonostante alcune di esse siano assai lunghe, non sono particolarmente adatte a buone misure di massa ed è quindi difficile basare su questo parametro una discussione degli eventi. La media delle masse di queste particelle risulta di $(1007 \pm 74) m_e$.

— PARTICELLE INSTABILI DI MASSA SUPERPROTONICA.

È ben stabilita attualmente l'esistenza di particelle instabili più pesanti dei protoni (particelle Y), per le quali sono stati finora proposti i seguenti schemi di decadimento:

$$(a) \quad \Lambda^0 \rightarrow p + \pi^- + 37 \text{ MeV}$$

$$(b) \quad \Sigma^\pm \rightarrow n + \pi^\pm + 110 \text{ MeV}$$

$$(c) \quad \Sigma^+ \rightarrow p + \pi^0 + 116 \text{ MeV}$$

$$(d) \quad \Xi^- \rightarrow \pi^- + \Lambda^0 + 60 \text{ MeV}.$$

I simboli che qui indicano i diversi iperoni sono gli stessi che impiegheremo nel testo.

Nella categoria degli iperoni possono essere anche inclusi certi tipi di frammenti instabili, considerati come nuclei contenenti una Λ^0 . Indicheremo nel seguito questi eventi con il simbolo F .

1) *Iperoni carichi*. — Nel corso di questa esperienza sono stati osservati 4 casi di decadimento di iperoni carichi, in moto ed a riposo, ed una cattura di iperone negativo. Discutiamo qui brevemente gli eventi e le loro possibili interpretazioni.

a) Decadimenti a riposo ($Y\text{-Pd}_2$ e $Y\text{-Pd}_5$). — La traccia dell'iperone $Y\text{-Pd}_2$ termina in emulsione e da essa prende origine una traccia assai scura, di aspetto protonico, che si arresta a sua volta dopo un percorso di 1,68 mm. La traccia primaria, lunga 0,95 mm, proviene da una piccola stella situata nella stessa pellicola in cui ha luogo il decadimento. Sia la traccia primaria che quella secondaria sono assai poco inclinate (angoli con il piano dell'emulsione: 4 e 5 gradi rispettivamente).

L'ipotesi che l'evento sia semplicemente lo scattering di un protone può essere scartata a causa delle notevoli discontinuità presentate dall' $\bar{\alpha}_{100}$ e dalla

ionizzazione specifica nel punto d'unione delle due tracce. La fig. 4 mostra l'andamento della densità di lacune in funzione del percorso misurato dal punto d'arresto della particella secondaria.

Le masse delle particelle primaria e secondaria sono state determinate dalle misure di scattering e ionizzazione in funzione del percorso. Nonostante la notevole incertezza sul valore delle masse, dovuta alla poca lunghezza delle tracce, siamo indotti a considerare questo evento come un esempio del decadimento di un Σ^+ secondo lo schema (c).

L'evento $Y-Pd_5$ è quasi identico al precedente, ma le tracce che lo formano sono più inclinate e quindi meno adatte a misure.

I valori di Q che risultano per questi eventi supponendo che per ambedue il decadimento avvenga se-

condo lo schema (c) sono rispettivamente di $116 \pm 1,5$ e $113 \pm 3,0$ MeV; valori notevolmente simili tra loro ed a quelli ottenuti da altri Autori⁽¹⁹⁾. Ci sembra che questa stretta somiglianza dei valori di Q costituisca una convincente conferma dello schema (c) con due soli prodotti di decadimento. Per lo stesso motivo pensiamo di poter escludere che gli eventi siano dovuti alla cattura di un mesone K^- o di un iperone negativo e successiva emissione di un protone, quantunque questa ipotesi sia compatibile, a rigore, con i dati sperimentali.

I dati relativi a questi eventi, ambedue identificati durante l'osservazione ordinaria, sono riportati in tab. IX.

b) Decadimenti in volo ($Y-Pd_3$ e $Y-Pd_4$). - L'evento $Y-Pd_3$, trovato durante la revisione delle stelle a due rami, è probabilmente, un esempio del decadimento in volo di un Σ^+ con emissione di un protone. Esso consiste di una traccia, uscente da stella, che dopo un percorso di 15 mm, subisce un improvviso cambiamento di densità (da $4,7 \pm 0,1 \times \text{min}$ a $3,7 \pm 0,1 \times \text{min}$) ed una deflessione angolare di 26 gradi. Nel punto di decadimento la traccia inverte il senso della sua inclinazione rispetto al piano della pellicola, conservando

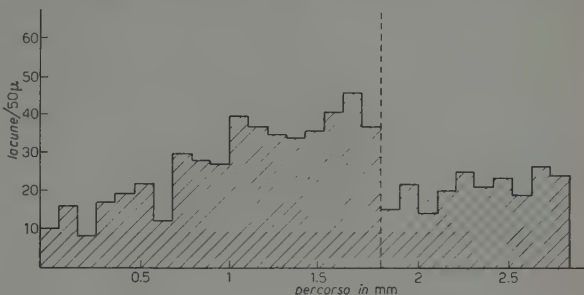


Fig. 4. - Variazioni della densità di lacune in funzione del percorso per le tracce formanti l'evento $Y-Pd_2$. La linea tratteggiata indica la discontinuità corrispondente al punto di decadimento.

⁽¹⁹⁾ P. ASTBURY, A. BONETTI, M. CECCARELLI, N. DALLAPORTA, C. FRANZINETTI, M. W. FRIEDLANDER and G. TOMASINI: *Suppl. Nuovo Cimento*, **12**, 448 (1954); A. BONETTI, R. LEVI SETTI, M. PANETTI and G. TOMASINI: *Nuovo Cimento*, **16**, 345 (1953); M. W. FRIEDLANDER, D. KEEFE and M. G. K. MENON: *Nuovo Cimento*, **1**, 482 (1955).

TABELLA IX.

| Evento | Stella di origine | PRIMARIO (*) | | | | SECONDARIO | | | | | Q | Osservatore |
|-------------------|-------------------|---------------|------------------------|--------------------------------------|-------------------------|-----------------|------------------------|--------------------------------------|------------------------|----------|-----------|-------------|
| | | Percorso (mm) | N. lastre attraversate | Massa (m _e) | t _r (s) | Percorso (mm) | N. lastre attraversate | Massa (m _e) | Energia cinetica (MeV) | Identità | | |
| Y-Pd ₂ | 5 + 1n | 0,95 (t) | 1 | 1840 ⁺⁹⁰⁰ ₋₆₀₀ | 2,9 · 10 ⁻¹¹ | 1,68 ± 0,03 (t) | 1 | 2200 ⁺⁶⁰⁰ ₋₄₀₀ | 18,8 ± 0,2 | p | 116 ± 1,5 | A. BERNARDI |
| Y-Pd ₅ | 3 + 0n | 2,3 (t) | 3 | ~ 2000 | 4,1 · 10 ⁻¹⁰ | 1,60 ± 0,05 (t) | 2 | — | 18,2 ± 0,4 | p | 113 ± 3,0 | A. BERNARDI |

TABELLA X.

| Evento | Stella d'ori- gine | PRIMARIO (*) | | | | | SECONDARIO | | | | | | | Q (MeV) | Osser- vatore | |
|-------------------|--------------------------|-----------------------|--------------------------------|--------------------------------------|-------------------------|-------------------------|----------------|------------------------------|--|------------------|-----------------------|--------------------------------|----------------------------|------------|--|-------------------|
| | | Per- corso (mm) | N. lastre attraver- sate | Massa (m _e) | t _r (s) | T _r (s) | β _d | θ | Energia cinetica (Mev) | I/I ₀ | Per- corso (mm) | N. lastre attraver- sate | Massa (m _e) | | | Identita |
| Y-Pd ₃ | 11 + 3n | 15,0 (o) | 8 | 1900 ± 250 | 1,4 · 10 ⁻¹⁰ | 3,7 · 10 ⁻¹⁰ | 0,371 | 26° S. L. 85° S. C. | 99,7 ± 1,2 S. L. 20,5 ± 6,1 S. C. | — | 28,7 ± 0,5 (t) | 10 | 2000 ± 300 | p | 125 ± 30 (Σ ⁺ → p + π ⁰) | B. VI- TALE |
| Y-Pd ₄ | 3 + 4n | 11,2 (o) | 11 | 2400 ⁺⁶⁰⁰ ₋₅₀₀ | 1,2 · 10 ⁻¹⁰ | ∞ | 0,32 | 15° S. L. 22° S. C. | — — | 1,05 ± 0,10 | 1,6 (o) | 2 | — | (π) | — | V. CHIA- RATTI |

(*) I simboli usati nelle Tabelle IX e X sono quelli indicati alle pagg. 459-460 dei Rendiconti del Congresso di Padova (Suppl. Nuovo Cimento, 2, (1954)).

però circa la stessa pendenza. Poichè la particella secondaria attraversa gli stessi strati di emulsione percorsi da quella primaria, riteniamo che la diminuzione di densità osservata sia dovuta veramente a una diminuzione di potere ionizzante e non invece a un gradiente di sviluppo.

La massa della particella primaria risulta circa protonica; possiamo quindi escludere che l'evento sia lo «stripping» di un deutone seguito dalla liberazione di un protone più veloce. Inoltre un simile processo porterebbe assai probabilmente a una deflessione della traccia e a un aumento di velocità molto minori di quelli osservati. Poichè la particella secondaria è un protone pensiamo che l'evento sia dovuto al decadimento in volo di un Σ^+ secondo lo schema (c). Il valore di Q calcolato in questa ipotesi, in buon accordo con quelli ottenuti nei decadimenti a riposo, risulta di 125 ± 30 MeV.

L'evento Y-Pd₄, trovato anch'esso durante la revisione delle stelle a due rami, è in condizioni geometriche tali da rendere difficili ed imprecise le misure.

La particella primaria proviene da una disintegrazione nucleare, percorre circa 1 mm in ogni lastra e decade in volo dopo 1,2 mm. La sua massa risulta di $2400^{+600}_{-400} m_e$.

La traccia secondaria è praticamente al minimo. Essa percorre 0,7 mm per lastra e non è seguibile con sicurezza nelle pellicole successive. Se l'evento fosse dovuto al decadimento di un Σ^\pm secondo lo schema (b) l'energia del secondario dovrebbe essere di circa 130 MeV nel sistema del laboratorio, e questa energia è compatibile con il valore della sua ionizzazione.

I dati relativi a questi due eventi sono riportati nella tab. X.

c) Cattura nucleare di un iperone negativo. — In questo evento, trovato durante la revisione dei mesoni σ , una particella pesante si arresta in emulsione dopo un percorso di 9,4 mm e produce una stella formata da un protone di 60 MeV e da due rami di pochi grani (vedi tab. XI).

La massa della particella primaria, determinata dai valori dello scattering e della densità di grani in funzione del percorso, è risultata rispettivamente di 1900 ± 400 e $2100 \pm 300 m_e$. Questi valori della massa ci fanno ritenere che l'evento sia dovuto alla cattura di un iperone negativo.

È da notare che in questo evento, come in altri dello stesso tipo finora noti ⁽²⁰⁾, l'energia visibile nella disintegrazione è piuttosto piccola rispetto al valore di 250 MeV corrispondente alla completa diseccitazione di un Σ^- .

2) Iperoni neutri (Λ^0). — Come risultato dell'inseguimento a ritroso delle tracce di 292 pioni negativi, sono stati trovati tre eventi probabilmente dovuti al decadimento di particelle Λ^0 . Gli eventi appaiono come stelle a due rami che risultano essere rispettivamente un π^- ed un probabile protone.

Il calcolo dei valori di Q è stato fatto con la formula di Friedlander e

⁽²⁰⁾ R. H. W. JOHNSTON e C. O'CEALLAIGH: *Nuovo Cimento*, **1**, 468 (1955).

TABELLA XI.

| Evento | PRIMARIO | | | STELLA DI CATTURA | | Osservatore |
|-----------|--------------------|---------------------|------------------|---|--|-------------------|
| | Massa (m_e) | Stella d'origine | Percorso (mm) | Identità ed energia dei rami ionizzanti della stella | Energia visi- bile (MeV) | |
| Y^-Pd_1 | $2\,030 \pm 300$ | $27 + 2p$ | 9,43 (t) | $\begin{matrix} p \\ 63 \pm 5 \text{ MeV} \end{matrix}$ | $\begin{matrix} Z \neq 1 \\ R < 2 \mu \end{matrix}$ $\begin{matrix} Z \geq 1 \\ R < 2 \mu \end{matrix}$ | 65 A. BERNARDI |

TABELLA XII.

| Evento | Rami della « V » | | Caratteristiche del decadimento | | | Massa della « V » (m_e) | Osservatore |
|-----------|---------------------------|---|------------------------------------|-------------------------|----------------------------|-----------------------------------|-------------|
| | Identità | Percorso (mm) | Energia (MeV) | φ | θ p (MeV/c) | | |
| V^0Pd_1 | $\pi^- (\sigma_3)$ p | $26,8 \pm 0,6 (t)$ $6,15 \pm 0,15 (t)$ | $42,9 \pm 1,0$ $40,1 \pm 1,0$ | $69^\circ \pm 2^\circ$ | 76° 342 | $32,8 \pm 2,0$ | M. BERNO |
| V^0Pd_2 | $\pi^- (\sigma_5)$ p | $2,83 \pm 0,15 (t)$ $1,89 \pm 0,1 (t)$ | $11,3 \pm 0,7$ $20,1 \pm 1,0$ | $103^\circ \pm 2^\circ$ | 12° 180 | $15,5 \pm 1,5$ | A. SALANDIN |
| V^0Pd_3 | $\pi^- (\sigma_2)$ p | $24,1 \pm 0,8 (t)$ $2,8 \pm 0,2 (t)$ | $40,1 \pm 1,0$ $25,1 \pm 1,0$ | $93^\circ \pm 2^\circ$ | $86^\circ 40'$ 280 | $39,5 \pm 1,5$ | M. BERNO |
| V^0Pd_4 | $\pi^- (\sigma_2)$ p | $6,1 \pm 0,1 (t)$ $12,9 \pm 0,4 (t)$ | $17,9 \pm 0,2$ $61,8 \pm 1,2$ | $130^\circ \pm 2^\circ$ | 90° 300 | $38,9 \pm 2,0$ | M. BERNO |

φ = angolo tra la traccia del π^- e quella del p S.L.; θ = angolo tra la direzione di moto della V^0 e del π^- S.C.; p = quantità di moto della V^0 all'atto del decadimento.

coll. ⁽²¹⁾, impiegando la relazione energia percorso riportata nel § 1. I dati relativi a questi eventi si trovano nella tab. XII.

Questi tre eventi hanno valori di Q di circa 37 MeV, compatibili cioè con quello considerato oggi il più attendibile nella disintegrazione della Λ^0 . Nella tabella è riportato un quarto evento per il quale il Q risulta di 15,5 MeV. Esso potrebbe essere dovuto al decadimento di un iperone neutro diverso dalla Λ^0 ⁽²²⁾ od anche esser solo il risultato di una disintegrazione nucleare.

3) *Frammenti instabili.* - Durante la normale osservazione sono stati trovati cinque esempi di disintegrazione ritardata di frammenti nucleari instabili. I dati relativi a questi eventi, indicati con i simboli F-Pd₁, F-Pd₃, F-Pd₄, F-Pd₆ ed F-Pd₇, sono riportati nella tab. XIII. Nella stessa tabella sono inoltre riportati due eventi dello stesso tipo (F-Pd₂ ed F-Pd₅), nei quali però la traccia primaria è troppo breve per essere identificata come frammento nucleare.

Il valore della carica dei frammenti F-Pd₁, F-Pd₃, F-Pd₆ ed F-Pd₇ è stato ottenuto con misure di densità dei raggi δ in funzione del percorso ⁽²³⁾ valendosi di tarature eseguite su tracce di mesoni μ e di particelle α . Per il frammento F-Pd₄ possiamo invece indicare soltanto il limite inferiore di Z , pari alla somma delle cariche elementari liberate nella disintegrazione.

Nei tre eventi F-Pd₁, F-Pd₄ ed F-Pd₇, non c'è equilibrio tra gli impulsi delle particelle cariche; non possiamo quindi stabilire degli schemi di disintegrazione, ma soltanto dire che gli eventi sono compatibili con il « decadimento non mesonico » di una Λ^0 incorporata nel frammento all'atto della sua formazione ⁽²⁴⁾. Invece per i due frammenti F-Pd₃ ed F-Pd₆, tra i cui prodotti di disintegrazione è presente un mesone σ , è possibile stabilire degli schemi di decadimento per i quali si annulli la risultante degli impulsi delle particelle cariche.

L'evento F-Pd₃ può essere ricondotto all'uno o all'altro degli schemi:

$$a) \quad {}^4\text{He}^* \rightarrow {}^3\text{He} + \pi^- + p + Q \quad \text{con } Q = 35,8 \pm 1,2 \text{ MeV}$$

$$b) \quad {}^6\text{He} \rightarrow {}^4\text{He} + \pi^- + d + Q \quad \text{con } Q = 36,7 \pm 1,2 \text{ MeV}$$

mentre per l'evento F-Pd₆ lo schema più attendibile è:

$${}^3\text{H} \rightarrow {}^3\text{He} + \pi^- + Q \quad \text{con } Q = 42,6 \pm 1,0 \text{ MeV}.$$

⁽²¹⁾ M. W. FRIEDLANDER, D. KEEFE, M. G. K. MENON and M. MERLIN: *Phil. Mag.*, **45**, 533 (1954).

⁽²²⁾ W. CHESTON and H. PRIMAKOFF: *Phys. Rev.*, **93**, 908 (1954); J. P. ASTBURY: *Suppl. Nuovo Cimento*, **12**, 317 (1954).

⁽²³⁾ S. O. C. SÖRENSEN: *Phil. Mag.*, **42**, 188 (1951).

⁽²⁴⁾ W. CHESTON and H. PRIMAKOFF: *Phys. Rev.*, **92**, 1537 (1953).

TABELLA XIII.

| FRAMMENTO | | | | | DISINTEGRAZIONE | | | | | | Osservatore |
|-------------------|--------------------|--------------------|------------------------------|---|-----------------------------|--|--|---|---|-----------------------|--|
| Evento | Stella d'origine | Percorso (μ) | Angolo di inclinazione (Dip) | t (s) | θ_p | n. ramo | 1 | 2 | 3 | 4 | Schema di decadimento |
| F-Pd ₁ | 10+7p | 170 \pm 5 | 14° | 4 \div 5 | 15° | Identità Percorso (μ) Energia(MeV) | p(d-t) 105 (t) 3,7 | p(d-t) 670 (t) 11 | p 1520 (t) 17,8 | p 19700(t) 79 | |
| F-Pd ₇ | 12+3p | 70 \pm 4 | 21° | 4 \div 7 | 51° | Identità Percorso (μ) Energia(MeV) | p(d-t) 400 (t) 8,2 | Rinculo 4 (t) — | | | |
| F-Pd ₄ | 20+3p | 65 \pm 5 | 62° | ≥ 3 | 63° | Identità Percorso (μ) Energia(MeV) | d 3000 (t) 34 | p(d-t) 75 (t) 3 | p(d-t) 2385 (t) 23 | | |
| F-Pd ₃ | 12+3p | 3380 \pm 70 | 36° | 2 | 87° | Identità Percorso (μ) Energia(MeV) | $\pi^- (\sigma_1)$ 17400(t) | p(d-t) 57 (t) 2,6 (p) 3,3 (d) | Rinculo 1(t) 0,25(³ He) 0,4 (⁴ He) | | ⁴ He* = p + π^- + ³ He + Q Q = 35,8 \pm 1,2 ⁶ He* = d + π^- + ⁴ He + Q Q = 36,7 \pm 1,2 |
| F-Pd ₆ | 16+2 $\frac{1}{2}$ | 2200 \pm 50 | 45° | 1 | 6 \cdot 10 ⁻¹¹ | Identità Percorso (μ) Energia(MeV) | $\pi^- (\sigma_1)$ 24300(t) 40,3 | Rinculo 8 (t) 2,5(³ He) | | | ³ H* = π^- + ³ He + Q Q = 42,6 \pm 1,0 |
| F-Pd ₂ | 11+1p | 9 \pm 2 | 57° | 1,4 \cdot 10 ⁻¹² (⁷ Li) | 130° | Identità Percorso (μ) Energia(MeV) | p 400 (t) 8,2 | d 1400 (t) 21,5 | p(d-t) 800 (t) 12 | | |
| F-Pd ₅ | 12+0p | 3 \pm 1 | — | — | 80° | Identità Percorso (μ) Energia(MeV) | p(d-t) 1750 (t) 19,3 | p(d-t) 8400(o) 48 \div 55 | p(d-t) 68 (t) 2,8 | Rinculo 7 (t) — | |

I valori di Q calcolati per questi due frammenti sono in ottimo accordo con quelli ottenuti da altri Autori per eventi simili ^(25,26) e si accordano con l'ipotesi che le due disintegrazioni siano il risultato del « decadimento mesonico » di una Λ^0 incorporata in un frammento nucleare.

Anche qui come in altri esempi analoghi la Λ^0 risulta meno legata al frammento di quanto lo sarebbe il neutrone che essa sostituisce. Infatti l'energia di legame della Λ^0 risulta di 1,7 MeV per il nucleo ${}^4\text{He}^*$ e di meno di 1 MeV per il nucleo ${}^3\text{H}^*$, mentre le corrispondenti energie di legame di un neutrone sono di 20 e di 6,3 MeV.

I due eventi (F-Pd_2 ed F-Pd_5) il cui primario è di brevissimo percorso potrebbero essere dovuti a frammenti od a nuclei di rinculo contenenti una Λ^0 ⁽²⁷⁾. Non possiamo però escludere che essi siano dovuti invece alla cattura nucleare di mesoni od iperoni negativi.

Dalle disintegrazioni in cui hanno origine i sette frammenti instabili non sono stati visti provenire mesoni pesanti od iperoni. In due casi invece l'emissione del frammento instabile è accompagnata da quella di un frammento pesante stabile, in direzione quasi opposta.

— PRODUZIONE DELLE PARTICELLE PESANTI INSTABILI.

Lo studio dei fenomeni connessi colla produzione dei mesoni pesanti e degli iperoni costituisce probabilmente una delle vie migliori per giungere a una descrizione esauriente di queste particelle. Un tale studio riesce però alquanto complesso se effettuato sulle particelle prodotte dalla radiazione cosmica in lastre nucleari, a causa della imperfetta conoscenza dei nuclei colpiti e delle particelle primarie.

Pensiamo tuttavia che abbia un certo interesse riportare alcuni dati relativi alle disintegrazioni in cui sono emesse particelle pesanti instabili ed indicare quali riteniamo siano le frequenze di produzione dei diversi tipi di particelle.

1) *Osservazioni sulle stelle d'origine.* -- Tutte le tracce delle particelle instabili descritte sono state seguite fino alla loro origine. In 34 casi esse sono state viste provenire da disintegrazioni di nuclei dell'emulsione; in 2 casi invece esse provengono dall'esterno. È singolare il fatto che ambedue le tracce provenienti dall'esterno siano dovute a mesoni K^- .

⁽²⁵⁾ R. D. HILL, E. O. SALANT, M. WIGDOFF, L. S. OSBORNE, A. PEVSNER, D. M. RITSON, J. CRUSSARD and W. D. WALKER: *Bull. Am. Phys. Soc.*, **29**, 60 (1954).

⁽²⁶⁾ A. BONETTI, R. LEVI SETTI, M. PANETTI, L. SCARSI and G. TOMASINI: *Nuovo Cimento*, **11**, 210, 330 (1954); A. DE BENEDETTI, C. M. GARELLI, L. TALLONE and L. VIGONE: *Nuovo Cimento*, **12**, 466 (1954); M. DANYSZ: comunicazione privata.

⁽²⁷⁾ M. GOLDBABER: *Phys. Rev.*, **92**, 1279 (1953).

Le stelle da cui provengono le particelle instabili sono state analizzate per determinarne le caratteristiche generali, quali il numero di rami sottili e fortemente ionizzanti, l'esistenza di un primario carico, ecc., e per stabilire l'eventuale presenza di altre particelle instabili tra i prodotti della disintegrazione. I dati delle nostre stelle sono insufficienti per considerazioni statistiche; sembra però che dall'insieme dei dati oggi noti possa esser suggerita l'esistenza di

qualche diversità tra le stelle d'origine dei vari tipi di particelle instabili.

A titolo puramente indicativo osserveremo, ad esempio, che le stelle d'origine dei mesoni σ_K attualmente noti hanno un numero medio di rami minore di quello delle stelle d'origine dei mesoni K che decadono spontaneamente e che le stelle da cui provengono i τ sembrano in media di energia maggiore di quelle da cui provengono i K_L . La realtà di questi fatti potrà essere confermata solo da statistiche più ampie; ci sembra però che una decisione al riguardo possa avere grande interesse, costituendo una delle poche vie per decidere sull'identità di mesoni pesanti con decadimento diverso.

Le molteplicità delle stelle d'origine qui studiate sono riportate sotto forma di spettro integrale in fig. 5 insieme a un analogo spettro relativo a stelle generiche prodotte in condizioni simili. Si può osservare che i due istogrammi assumono praticamente la stessa pendenza per le molteplicità superiori a 5 rami sottili. Ciò indicherebbe che la produzione delle particelle pesanti instabili di bassa energia non aumenta sensibilmente con l'energia del primario quando questa sia superiore a qualche GeV.

Nel tentativo di mettere in evidenza l'eventuale produzione associata o in coppia dei mesoni K e degli iperoni ⁽²⁸⁾ abbiamo seguito per tutta la loro lunghezza il 60% dei rami grigi e neri provenienti dalle stelle d'origine dei mesoni K , degli iperoni e dei frammenti. Nessuna di queste tracce è risultata però dovuta a una seconda particella pesante instabile.

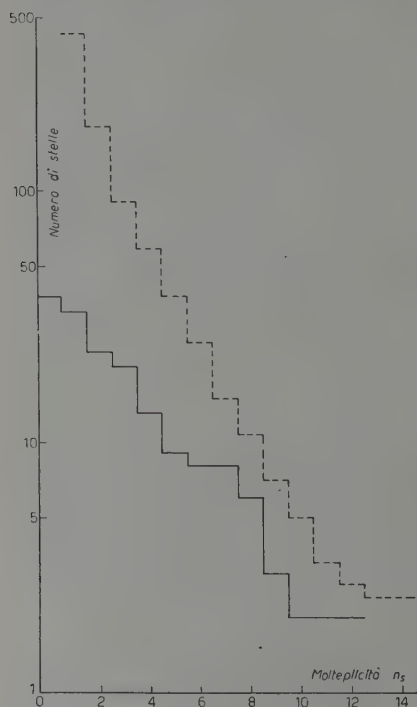


Fig. 5. — Spettro integrale della molteplicità di stelle generiche prodotte dalla radiazione cosmica (---) e delle stelle d'origine delle particelle pesanti instabili (—). Le due distribuzioni si riferiscono a volumi diversi di emulsione.

⁽²⁸⁾ Per l'intera bibliografia vedi: K. GOTTSTEIN: *Nuovo Cimento*, **1**, 284 (1955).

2) *Frequenza d'osservazione dei mesoni pesanti.* — Il numero effettivo di eventi di un certo tipo presenti in un cm^3 d'emulsione nucleare può essere determinato solo a condizione che si possa valutare il coefficiente di perdita dell'osservazione al microscopio, cioè il rapporto tra il numero d'eventi che sfuggono o che non sono identificabili e il numero di quelli realmente presenti. Tale coefficiente può assumere valori notevolmente diversi per differenti tipi di eventi, tanto che in certi casi può essere trascurabile ed in altri essere prossimo all'unità.

Mentre nel caso in cui gli eventi siano piuttosto numerosi è possibile valutare la perdita dovuta all'osservazione ripetendo questa più volte od effettuandola in modo molto lento e con forte ingrandimento, ciò non è possibile per eventi molto rari, perchè una ricerca lenta condurrebbe ad un numero di particelle troppo piccolo per avere significato statistico. Abbiamo cercato perciò di risolvere il problema in modo indiretto esaminando i diversi motivi che possono portare alla perdita di un evento e cercando di ricondurre i fattori di perdita degli eventi rari a quelli, più facilmente determinabili, di eventi più frequenti. Riportiamo qui solo una traccia del metodo impiegato riservandoci di parlarne in altra occasione.

Ogni evento di un certo tipo potrà essere considerato, agli effetti della sua identificazione da parte dell'osservatore, come la risultante di diverse caratteristiche, ognuna delle quali ha una diversa probabilità di venire osservata. Per esempio l'evento « decadimento a riposo di un mesone K_1 » sarà riconosciuto come tale solo quando si osservi contemporaneamente: a) la presenza di una traccia scura, b) la presenza di una traccia sottile ad un suo estremo, c) che questo estremo è un punto d'arresto e non di origine e d) che la massa della particella primaria è superiore a quella di un mesone μ . Nel caso in cui non venga osservata la prima caratteristica l'evento sfuggirà completamente; se invece non risultano osservate rispettivamente la seconda, la terza o la quarta, l'evento sarà classificato come protone, stella a 2 rami o evento $\mu \rightarrow e$.

Per riconoscere, invece, l'evento « cattura nucleare di un mesone K^- » sarà necessario osservare: a) una stella piccola, b) che uno dei rami si arresta nel centro della stella e c) che questo ramo non è dovuto a un mesone π . Quando manchi l'osservazione b), l'evento sarà considerato una stella piccola e invece una stella di σ quando manchi c). Può verificarsi però che tra i rami della stella di cattura ve ne sia uno sottile; in questo caso l'evento σ_K potrà essere identificato correttamente anche mancando l'osservazione c).

Questa analisi potrebbe essere ripetuta per la maggior parte degli eventi che a noi interessano.

È da notare che l'osservare alcune delle caratteristiche è assolutamente indispensabile affinché l'evento venga trovato, mentre se mancano le altre l'evento sarà classificato erroneamente ma avrà ancora la possibilità di essere identificato nel corso di eventuali revisioni. Anche alle revisioni è poi natu-

ralmente necessario attribuire un certo rendimento, di cui si dovrà tener conto nel calcolare il numero effettivo degli eventi.

Per valutare la probabilità di osservare una data caratteristica dovremo dapprima supporre che sia ugualmente probabile accorgersi di essa in un evento raro o in un evento comune. Potremo allora cercare una categoria di eventi, abbastanza comuni, che abbiano una caratteristica uguale a quella che vogliamo vedere nell'evento raro, e determineremo la probabilità di osservare in essi la caratteristica in questione paragonando tra loro i risultati della osservazione ordinaria con quelli di una ricerca molto lenta ed eventualmente ripetuta.

Ad esempio, potremo ricondurre la probabilità di osservare un mesone K_L (caratteristica a) a quella di osservare un protone, e la probabilità di vedere il suo secondario al minimo (caratteristica b) a quella di vedere l'elettrone di un mesone μ .

Sarà inoltre possibile collegare tra di loro caratteristiche simili di eventi diversi: considerare, ad esempio, che la probabilità di confondere il mesone K con un mesone leggero sia uguale negli eventi K_L e σ_K . Naturalmente la probabilità di osservare la traccia secondaria di un mesone K_L aumenterà al crescere della sua densità di grani. Ad esempio, abbiamo potuto constatare dallo studio dei coefficienti di perdita dei rami sottili uscenti da stelle, che i secondari dei mesoni $K_{\pi 2}$ devono essere in genere alquanto più visibili di quelli degli altri K_L .

Determinate così le probabilità parziali di osservazione, la probabilità di identificare un evento di un certo tipo si ottiene infine dal prodotto delle probabilità di osservazione delle singole sue caratteristiche, nell'ipotesi che esse siano indipendenti tra loro.

In alcuni casi non è però possibile considerare le diverse probabilità come completamente indipendenti; per esempio, la probabilità di non riconoscere il senso di moto del primario di una σ_K cresce con il numero di rami della disintegrazione. In altri casi non è invece possibile determinare la probabilità di una caratteristica perchè non esistono eventi comuni adatti per il confronto. Sarà meglio allora procedere alla revisione accurata di tutti gli eventi con cui possono essere stati confusi quelli desiderati e rinunciare a valutare le probabilità più incerte.

Riteniamo che il metodo qui brevemente descritto possa condurre a risultati abbastanza attendibili per quasi tutti i tipi di eventi, purchè naturalmente la determinazione delle probabilità di osservazione sia fatta di volta in volta in relazione al tipo di emulsioni usate e ad ogni gruppo di osservatori. Forse soltanto nel caso dei mesoni K_L , che sono spesso eventi al limite della visibilità, possono intervenire in modo importante dei fattori psicologici di cui è difficilissimo tenere conto.

Per questo motivo abbiamo cercato di porre a confronto la frequenza di osservazione di queste ultime particelle coi risultati dell'esperienza di DAHA-

NAYAKE e coll. ⁽²⁹⁾, estrapolati a stelle più piccole e a mesoni K_l più lenti di quelli da loro considerati. Abbiamo cioè calcolato il numero di eventi K_l che sarebbe stato trovato da quegli Autori qualora essi avessero esteso le misure di massa a tutte le tracce presenti nel blocco di emulsione. Poichè le lastre da essi usate hanno avuto una esposizione simile alle nostre, possiamo paragonare la loro frequenza di mesoni K_l con quella che risulta dai nostri dati corretti nel modo descritto. L'accordo tra i due valori risulta abbastanza soddisfacente.

TABELLA XIV.

| | K_l | $K_{\pi 2}$ | τ | K^- |
|--|-----------|-------------|--------|-------|
| Numero di eventi osservati. | 12 | 3 | 6 | 6 |
| Rendimento dell'osservazione | 30 % | (15 %) | 80 % | 70 % |
| Numero di eventi/cm ³ .giorno | 2,0 ÷ 3,0 | (0,7 ÷ 1,0) | 0,4 | 0,4 |

Nella tab. XIV è riportato il numero dei mesoni pesanti di diverso tipo trovati nel corso di questa esperienza, le corrispondenti percentuali di perdita ed il numero di eventi che riteniamo siano effettivamente presenti nell'unità di volume. I valori sottolineati possono essere considerati piuttosto attendibili mentre quelli tra parentesi sono soltanto indicativi. Nel caso dei mesoni $K_{\pi 2}$ la percentuale di perdita è stata calcolata tenendo presente che le tracce secondarie dovevano essere lunghe almeno 4 mm per lastra per poter essere identificate.

3. - Conclusioni.

I dati riportati in questo lavoro, relativi a molti tipi diversi di eventi, mentre nel loro insieme costituiscono forse un contributo statistico di un certo peso alle ricerche sui mesoni pesanti non sempre si prestano ad una discussione isolata. Preferiamo perciò limitarci a quelle conclusioni che si possono trarre direttamente da essi e rimandare invece negli altri casi il lettore alle discussioni finali dei Congressi di Padova e di Rochester ⁽³⁰⁾ ed ai resoconti di carattere riassuntivo ⁽³¹⁾.

⁽²⁹⁾ C. DAHANAYAKE, P. E. FRANCOIS, Y. FUJIMOTO, P. IREDALE, C. J. WAD-
DINGTON and M. YASIN: *Phil. Mag.*, **45**, 855 (1954).

⁽³⁰⁾ Relazioni dei comitati speciali: *Suppl. Nuovo Cimento*, **12**, 419-473 (1954);
Proc. Rochester Confer. 1955, in corso di pubblicazione.

⁽³¹⁾ B. ROSSI: *Rendiconti Scuola di Varenna* 1954, in corso di pubblicazione;
G. P. S. OCCHIALINI: *Ann. Rev., Nucl. Sc.*, in corso di pubblicazione nel vol. IV.

Mesoni K. – Crediamo di aver ottenuto con questa esperienza una verifica piuttosto convincente dell'esistenza di una particella instabile con una massa di $953 \pm 30 m_e$, che decade secondo lo schema:

$$K_{\pi 2}^{\pm} \rightarrow \pi^{\pm} + \pi^0 + 213 \text{ MeV}.$$

La massa di questo mesone è assai prossima a quella della particella θ^0 ⁽³²⁾, della quale esso potrebbe essere il corrispondente carico.

La massa del mesone $K_{\pi 2}$ è anche molto prossima a quella del mesone τ tanto da suggerire l'ipotesi che questi due tipi di eventi corrispondano soltanto a modi diversi di decadimento di una stessa particella. Anche però nel caso in cui questa ipotesi non potesse esser sostenuta per la difficoltà di trovare comuni valori per lo spin e la parità che siano compatibili coi dati sperimentali ⁽³³⁾, riteniamo che l'esistenza di due bosoni di massa quasi identica e dalle caratteristiche abbastanza simili possa difficilmente esser considerata puramente casuale.

Mentre per i mesoni $K_{\pi 2}$ e τ i valori della massa possono essere ricavati da quelli del ϱ e sono perciò affetti da errori piuttosto piccoli, la massa degli altri mesoni K, dovendo essere misurata direttamente, è conosciuta con minor esattezza e non costituisce quindi un parametro molto significativo per paragonare tra loro eventi di tipo diverso. I risultati di misure dirette di masse effettuate sui mesoni K_L e K^- di questa esperienza sono in ogni modo compatibili con l'ipotesi che tutte queste particelle abbiano una massa assai vicina a quella del mesone τ , od almeno che siano piuttosto rare le particelle sensibilmente più pesanti di $1000 m_e$.

Le frequenze degli eventi di diversi tipi, riportate nella Tab. XIV mostrano che i mesoni pesanti che decadono spontaneamente sono molto più numerosi di quelli che subiscono un processo di cattura nucleare. Se supponiamo che la maggior parte dei mesoni K negativi dia luogo ad una stella di cattura visibile, avremo che il rapporto tra il numero dei mesoni che decadono spontaneamente e di quelli che sono catturati è uguale al rapporto tra le particelle positive e negative. Secondo i nostri dati i mesoni K positivi sarebbero allora circa 10 volte più numerosi dei negativi.

I risultati di alcune esperienze effettuate con camere di Wilson ⁽³⁴⁾ sembrano suggerire che i mesoni K^- possono essere i corrispondenti negativi dei τ e forse dei $K_{\pi 2}$ e $K_{\mu 3}$, ma invece escludere che essi siano corrispondenti dei mesoni $K_{\mu 2}$ che sarebbero invece prodotti quasi senza eccezione nello stato di carica posi-

⁽³²⁾ R. W. THOMPSON, J. R. BRUWELL, O. H. COHN, R. W. HUGGET, C. J. KARZMARK and Y. B. KIM: *Proc. Rochester Conf.*, 1954.

⁽³³⁾ E. FABRI: *Nuovo Cimento*, **11**, 479 (1954).

⁽³⁴⁾ R. ARMENTEROS, B. GREGORY, A. LAGARRIGUE, L. LEPRINCE-RINGUET, F. MULLER and CH. PEYROU: *Suppl. Nuovo Cimento*, **12**, 324 (1954).

tiva. Accettando questa ipotesi dovremo eliminare la frazione di mesoni $K_{\mu 2}$ dal totale dei K_1 prima di confrontare tra loro il numero delle particelle positive e negative, per evitare che il confronto avvenga tra gruppi non omogenei di particelle.

Dalla forma dello spettro energetico dei secondari ⁽³⁵⁾, tenendo conto dei fattori di perdita che incidono maggiormente sulle tracce più sottili, si può stimare che i mesoni $K_{\mu 2}$ siano circa la metà del totale dei K_1 . Ne risulterebbe allora che i mesoni pesanti diversi dai $K_{\mu 2}$ sono prodotti nello stato di carica positiva con frequenza circa 5 volte maggiore che in quello di carica negativa.

L'eccesso positivo di questa frazione di mesoni pesanti non sarebbe quindi tanto elevato da non poter essere giustificato, accettando l'ipotesi della produzione associata ⁽³⁶⁾, con delle considerazioni sul peso statistico dei vari stati di carica analoghe a quelle usate per calcolare l'eccesso positivo dei mesoni leggeri ⁽³⁷⁾.

La percentuale dei mesoni K^- che risulta dalla nostra esperienza è sensibilmente superiore a quella ottenuta in molti altri laboratori. Questa discordanza può derivare sia dall'impiego di metodi di osservazione meno efficienti del nostro che da differenze nelle dimensioni dei blocchi di emulsione e nelle modalità di esposizione. La discussione di questi effetti sarebbe molto interessante ma ci sembra ancora prematura.

Concludiamo queste note sui mesoni K con una osservazione sulla misura della loro vita media.

Alcuni Autori ⁽³⁸⁾ hanno calcolato la vita media di queste particelle dal rapporto tra il numero di quelle che si arrestano in emulsione e quelle che invece disintegrano in volo. Ci sembra però che questo rapporto non possa essere direttamente ricavato dai dati dell'osservazione perchè, mentre il fattore di perdita degli eventi del primo tipo è in generale abbastanza piccolo, esso potrà raggiungere valori elevati per l'osservazione dei decadimenti in volo.

Ad esempio, l'evento da noi interpretato come decadimento in volo di un mesone τ non avrebbe potuto essere riconosciuto, nonostante l'aspetto piuttosto tipico del decadimento e la piccola velocità del primario, se non avessimo determinato sistematicamente il senso di moto di tutti i rami scuri delle stelle piccole. Si aggiunga poi che i decadimenti di particelle instabili piuttosto veloci hanno una grande probabilità di sfuggire ad una revisione limitata solamente ai rami più ionizzanti di ogni stella.

Poichè non ci risulta che sia stato ancora tentato un lavoro di revisione

⁽³⁵⁾ C. C. DILWORTH, A. MANFREDINI, G. D. ROCHESTER, J. WADDINGTON and, G. T. ZORN: *Suppl. Nuovo Cimento*, **12**, 433 (1954).

⁽³⁶⁾ A. PAIS and M. GELL-MANN: *Proceedings of the 1954 Glasgow Conference*, in corso di pubblicazione.

⁽³⁷⁾ P. CALDIROLA: *Nuovo Cimento*, **6**, 565 (1949).

⁽³⁸⁾ E. AMALDI: *Rendiconti Scuola di Varenna 1954*, in corso di pubblicazione.

di stelle che possa garantire una discreta efficienza nell'identificazione dei decadimenti in volo pensiamo che i valori della vita media che si ottengono applicando il metodo citato a tutti gli eventi finora noti siano alquanto maggiori del vero.

Una valutazione della vita media dei mesoni τ , ottenuta dall'insieme dei soli nostri dati, porta ad un valore dell'ordine di 10^{-9} secondi.

Iperoni. — Nel corso di questa ricerca sono stati trovati 3 eventi che confermano in modo molto convincente l'esistenza di un iperone positivo che decade secondo lo schema:

$$\Sigma^+ \rightarrow p + \pi^0 + 115 \text{ MeV}.$$

Questa particella può probabilmente decadere anche in un pione positivo ed un neutrone; non esiste però ancora una prova certa di questo fatto.

La massa della particella Σ^+ risulta dai nostri dati di $2325 \pm 3 m_e$. La sua vita media è probabilmente inferiore a quella del mesone τ e dev'essere dell'ordine di 10^{-10} secondi.

L'eccesso positivo degli iperoni, considerato uguale al rapporto tra il numero delle particelle che si arrestano e decadono spontaneamente e quello delle particelle catturate in modo visibile, sembra essere di poco superiore all'unità. Esso potrebbe però essere ancora più piccolo se si supponesse che numerosi iperoni negativi non vengono osservati perchè danno luogo a stelle di cattura da cui escono solamente delle particelle neutre.

Nel corso del lavoro abbiamo raccolto alcuni dati relativi ai fenomeni di cattura di una Λ^0 all'interno di un frammento nucleare. Per una discussione approfondita di questi fenomeni rimandiamo ai resoconti di GRILLI e LEVI SETTI ⁽³⁹⁾ e di LEVI SETTI ⁽⁴⁰⁾. Ci limitiamo a ricordare due degli eventi descritti in precedenza i quali si interpretano perfettamente supponendo che una Λ^0 possa sostituire uno dei neutroni di un nucleo leggero e che essa sia legata al nucleo con un'energia molto minore di quella con cui dovrebbe esser legato il corrispondente neutrone.

Ringraziamenti.

Desideriamo al termine di questo lavoro ringraziare coloro che l'hanno reso possibile, ed in primo luogo il prof. A. ROSTAGNI, direttore di questo Istituto, che ha sempre seguito la nostra ricerca con grande interesse fornendoci

⁽³⁹⁾ M. GRILLI and R. LEVI SETTI: *Suppl. Nuovo Cimento*, **12**, 466 (1954).

⁽⁴⁰⁾ R. LEVI SETTI: *Rendiconti Scuola di Varenna* 1954, in corso di pubblicazione.

preziose raccomandazioni ed i proff. N. DALLAPORTA ed M. MERLIN per il costante apporto di discussione e consigli. Desideriamo inoltre ringraziare ancora quest'ultimo insieme ai dott. G. QUARENI e I. SCOTONI ed ai colleghi delle Università di Bristol, Milano e Roma per aver collaborato all'esposizione ed allo sviluppo del materiale fotografico impiegato nella presente ricerca. Siamo lieti di potere in questa sede ricordare anche tutti coloro che hanno con tanto entusiasmo contribuito al lancio ed al ricupero dei palloni stratosferici: il personale degli Aeroporti di Elmas, Capodichino e Padova ed in particolare il personale dei rispettivi uffici meteorologici; i Comandanti e gli equipaggi delle navi « Altair », « Bracco » e « Pomona » e degli aerei di soccorso di Elmas e tutti gli altri i cui nomi non possono purtroppo per mancanza di spazio essere qui riportati.

Siamo infine grati a tutto il gruppo di osservatori del nostro Istituto per la collaborazione intelligente nell'esplorazione delle emulsioni; ed in particolare ai sigg. A. BERNARDI, M. BERNO, V. CHIARATTI, F. PROCOPIO e al laureando G. SALANDIN per l'aiuto datoci nei lavori di revisione e di taratura.

SUMMARY

During the past year a systematic investigation of heavy unstable particles has been carried out at the University of Padua using two stacks of 40 Ilford G5 nuclear research stripped emulsions each 600μ thick, exposed at balloon altitudes. In Table III is given a summary of the events observed during the first scanning as well as those found during the systematic re-examination. Later a detailed analysis of all unstable particle events was made. For particles whose ranges terminated in the emulsion, the mass values were obtained from scattering measurements applying both the method of « constant sagitta » and a method similar to the « P » method of the Bristol group, as well as the method of « comparative grain densities » based on measurements of specific ionization. The masses of those particles which did not come to the end of their ranges in the emulsion were determined from measurements of multiple scattering and specific ionization. These mass values were based on calibration curves determined for particles of known mass and charge. 12 K_1 mesons decaying at rest have been observed (Tab. IV). An average mass value of $985 \pm 30 m_0$ has been obtained and the mass distribution about this mean is consistent with a unique mass value for all K_1 particles observed. In four of the events the minimum secondary particles were emitted at very small angles with respect to the emulsion plane. In Fig. 1 is shown a $p\beta$ vs. g^* plot of the four K-meson secondaries and of 20 long π -meson tracks. It would seem that 3 of the events listed in the table could be satisfactorily explained by the scheme: $K_{\pi 2}^{\pm} \rightarrow \pi^{\pm} + \pi^0 + Q$ in that the 3 secondaries seem to be very probably π -mesons, each of the same energy. Using for the $p\beta$ of the secondary particle the value of 163 ± 7 MeV/c, and for the primary mass a value of $962 \pm 38 m_0$ and applying the momentum laws, we have calculated the mass of the neutral secondary to be

$296 \pm 60 m_e$. Six τ -mesons decaying at rest have been observed in this investigation; three of the τ -events are « complete » and were produced by a positively charged primary. The average value for the Q is 75.4 ± 0.4 MeV which corresponds to a mass of $967 \pm 0.8 m_e$ (see table V). In another event a K-particle comes to rest in the emulsion layers and decays into a single positive pion of 6 MeV. This event has also been interpreted as the decay of a τ -meson, in which the τ -decays by an alternative mode into a charged pion and two neutral particles. In addition an event was found during the re-examination of « little stars », which was very probably due to the disintegration in flight of a τ -meson (see photograph 1 and Table VI). Six events attributed to the capture of K-mesons have been observed (see Table VIII). The weighted average of measurements of the masses of the primaries is $1007 \pm 74 m_e$; which is very close to that of the K_1 . In the search for Λ^0 , 292 negative pions were followed back, four of them being observed to come from stars in which only one other charged particle was involved. Three of these two prong stars (see Table XII) seem to have been produced by the decay in flight of a neutral hyperon according to the scheme: $\Lambda^0 \rightarrow \pi^- + p + 37$ MeV. Four decays of charged hyperons have been observed. Two of the « primary » particles come to rest in the emulsion, both decaying into a low energy proton; the energies of these protons being 18.8 ± 0.2 MeV and 18.2 ± 0.4 MeV respectively. In the remaining two events the « primaries » decay in flight; one into a proton and the other into a pion. In another event a particle of near-protonic mass comes to the end of its range and is apparently captured by a nucleus of the emulsion producing a σ_V -star (see Tables IX, X and XI). Five clear examples of the delayed disintegration of a heavy fragment emitted from a nuclear disintegration have been observed. In one of these events the disintegration of a He-nucleus is accompanied by the emission of a negative pion and has been interpreted as being the result of the mesonic decay of a Λ^0 -particle, whose binding energy has been calculated to be 1.7 MeV. In another event we are dealing with the disintegration of a probable Tritium nucleus, with the emission of a 40.3 MeV π -meson and a 2.5 MeV He-nucleus. For this event the binding energy of the Λ^0 is found to be less than 1 MeV. The stars from which K-particles are emitted generally exhibit a rather low multiplicity: we are therefore led to the conclusion that K-meson production at « primary » energies of less than 5 GeV is an important phenomenon. We have no examples of the double emission of heavy unstable particles. By making an analysis of the sources of loss during the scanning, we have tried to estimate the frequencies of the various types of events. These frequencies, together with the proposed observational efficiencies for each class of event, are presented in Table XIV.

New Method for the Solution of the Deuteron Problem, and its Application to a Regular Potential.

F. CAP

Institute of Theoretical Physics of Innsbruck University - Innsbruck, Austria

W. GRÖBNER

Mathematical Seminary of Innsbruck University, and Research Center Innsbruck, Austria

(ricevuto il 18 Aprile 1955)

Summary. — First, success and failure of the usual linear charged PS meson theory of nuclear forces are discussed and reasons given for non-linear terms, yielding finite potential. Second, a new method of solution of the deuteron problem is given and applied to a regular PS potential. It is shown that the discussed non-linear PS theory yields far better results for the deuteron than the linear theory.

1. — Introduction.

In the theory of nuclear forces we are facing a principal difficulty: On the one hand, we have to assume with certainty that the nuclear forces are carried by pions ⁽¹⁾, and that these mesons are pseudoscalar ⁽²⁾; on the other hand, the usual (linear) pseudoscalar meson theory is not able to explain, except for the scattering data of low energy, any experimental data ⁽³⁾. Although it has been stated for several times that also the deuteron could

⁽¹⁾ J. HEIDMANN: *Journ. Phys. et Rad.*, **14**, 481 (1953); R. MOORHOUSE: *Adv. in Phys.*, **2**, 185 (1953); BURFENING *et al.*: *Phys. Rev.*, **75**, 382 (1949); W. J. BOLDANSKIJ *et al.*: *Usp. fiz. Nauk SSSR*, **48**, 531 (1952); G. EDER: *Acta Phys. Austr.*, **8**, 234 (1954).

⁽²⁾ J. GUNN: *Adv. in Phys.*, **2**, 213 (1953); E. HENLEY: *Phys. Rev.*, **87**, 42 (1952); R. MARSHAK: *Meson Physics* (New York, 1952); *Rev. Mod. Phys.*, **23**, 137 (1952); G. LÜDERS *et al.*: *Zeits. f. Naturj.*, **7a**, 213 (1952); J. ŠMUŠKEVIČ: *Žurnal eksper. teor. fiz.*, **22**, 251 (1952).

⁽³⁾ See items 2 and 28 in the EOARDC report 633-C or HULL *et al.*: *Phys. Rev.*, **90**, 482 (1953); KOMODA: *Progr. Theor. Phys.*, **9**, 468 (1953); **11**, 11 (1954).

be explained by the potentials of second and fourth order of the charge symmetrical PS theory ⁽⁴⁾, this is *not* correct according to most recent exact computations and checkups ⁽⁵⁾. As the singularity does not lead to any bound state, it previously simply has been cut-off ⁽⁶⁾; today — again phenomenologically — a « hard core » is introduced ⁽⁷⁾, the field theoretical proof of which has been attempted afterwards ⁽⁸⁾. But also this potential, which is infinite into the « other sense », was not in a position ⁽⁵⁾ to explain the deuteron problem. The rather good result obtained by LÉVY ⁽⁴⁾ probably has to be explained by the circumstance that he had at his disposal two free parameters (coupling constant g and core radius r_c) for the interpretation of essentially three quantities, of the binding energy and of the integrals (1) and (2)

$$(1) \quad Q = \frac{1}{5\sqrt{2}} \int_0^\infty r^2 uv \, dr - \frac{1}{20} \int_0^\infty r^2 v^2 \, dr,$$

$$(2) \quad \mu_D = \mu_N + \mu_P - \frac{3}{2} \left(\mu_N + \mu_P - \frac{1}{2} \right) \int_0^\infty v^2 \, dr,$$

and that $r_c = 0.53 \cdot 10^{-13} \text{ cm} = 0.38 (1/\kappa)$ was chosen in such a way that these integrals yielded the experimental values already in the range of $r_c \leq r \leq \infty$. (As it is known, the wave function vanishes for $r < r_c$, see e.g. MOORHOUSE).

Therefore it seemed to be of interest — after the singular potentials of both kinds had failed to explain the experimental deuteron data within the framework of a pseudoscalar meson theory — to attempt a solution of this problem

⁽⁴⁾ M. TAKETANI: *Progr. Theor. Phys.*, **6**, 638 (1951) and **6**, 635 (1951); S. MACHIDA and S. ONUMA: *Progr. Theor. Phys.*, **6**, 904 (1951); M. TAKETANI *et al.*: *Progr. Theor. Phys.*, **6**, 581 (1951); R. JASTROW: *Phys. Rev.*, **91**, 749 (1953); M. LÉVY: *Phys. Rev.*, **88**, 725 (1952), but see *Progr. Theor. Phys.*, **9**, 555 (1953); E. ARAKI and Y. MORI: *Progr. Theor. Phys.*, **6**, 188 (1951); M. LÉVY: *Compt. Rend.*, **234**, 1744 (1946 (1952)).

⁽⁵⁾ J. BLATT and M. KALOS: *Phys. Rev.*, **92**, 1563 (1953); M. LÉVY: *Phys. Rev.*, **84**, 441 (1951) and MOORHOUSE (1953).

⁽⁶⁾ H. BETHE: *Phys. Rev.*, **57**, 390 (1940); SATO: *Progr. Theor. Phys.*, **10**, 323 (1953); L. GOLDFARB *et al.*: *Phys. Rev.*, **88**, 1099 (1952).

⁽⁷⁾ R. JASTROW: *Phys. Rev.*, **79**, 389 (1950); HULTÉN: *Ark. Mat. Astr. Fys.*, **31-A**, No. 15, 11 pp. (1945).

⁽⁸⁾ K. BRUECKNER and F. LOW: *Phys. Rev.*, **83**, 461 (1951); R. JASTROW: *Phys. Rev.*, **81**, 165 (1950); **81**, 636 (1951); ENATSU and TAKANO: *Progr. Theor. Phys.*, **4**, 543 (1949); J. SATO: *Progr. Theor. Phys.*, **10**, 323 (1953); K. BRUECKNER and K. WATSON: *Phys. Rev.*, **92**, 1023 (1953); S. MACHIDA and K. NISHIJIMA: *Progr. Theor. Phys.*, **7**, 57 (1952), not favouring the core; P. OLSSON: *Ark. f. Fys.*, **5**, 122 (1952); J. SATO: *Progr. Theor. Phys.*, **10**, 323 (1953); P. OLSSON: *Phys. Rev.*, **83**, 845 (1951); N. AUSTERN: *Phys. Rev.*, **88**, 1207 (1952); E. SALPETER: *Phys. Rev.*, **91**, 994 (1953); PEASE and FESHBACH: *Phys. Rev.*, **81**, 142 (1951).

by means of *regular* potentials. Because such potentials have to go over into the potential of the linear charge symmetrical pseudo scalar theory at the latest from $1/\kappa = 1.40 \cdot 10^{-13}$ cm (from $m_\pi = 276 m_e$) on, it seemed to be of advantage to start out directly from a regular PS potential. As the linear PS theory yields a singular potential at $r = 0$ (r^{-3} term or infinite hard core) it was necessary to consider a *non-linear* PS theory.

By the way, a number of other reasons are in favour of non-linear terms in the meson field equations, such as:

1) The singular potentials should not be a result of the quantum theoretical perturbation computation, but should rather — at least partly — be caused by the shape of the field equations ⁽⁹⁾; PAULI and BETHE have shown that exactly the same singularities are to be found also at the classical derivation of nuclear forces ⁽¹⁰⁾.

2) The rest energy of the nucleon and therefore the energy of the meson field associated with it are finite in nature. It does not seem to be possible to obtain finite values for the field energy of point sources within the framework of linear field theories ⁽¹¹⁾.

3) The impossibility to renormalize the linear PS_{pv} theory ⁽¹²⁾.

4) The fact that the hard core, although existing, probably is not infinite but rather finite ^(7, 8).

5) Non-linear shielding and saturation by non-linear terms ⁽¹³⁾; the transparency of nuclei.

6) Non-linear vacuum polarization automatically producing a $\lambda\varphi^3$ -term

⁽⁹⁾ J. RAYSKI: *Acta Phys. Pol.*, **11**, 314 (1952); *Nuovo Cimento*, **10**, 1, (1953).

⁽¹⁰⁾ W. PAULI: *Meson Theory of Nuclear Forces* (New York, 1946); H. BETHE: *Phys. Rev.*, **57**, 260 (1940); BHABHA: *Proc. Roy. Soc.*, O **172**, 348 (1939); HARISH CHANDRA: *Proc. Roy. Soc.*, A **185**, 269 (1946); M. FIERZ: *Helv. Phys. Acta*, **14**, 257 (1941); LECOUEUR: *Proc. Camb. Phil. Soc.*, **45**, 429 (1949); N. SIL: *Ind. Journ. Phys.*, **27**, 311 (1953) and MARSHAK (1952).

⁽¹¹⁾ H. STEINWEDEL: *Fortschr. d. Phys.*, **1**, 7 (1953); H. STEINWEDEL: *Sitzungsber. Heidelb. Akad. Math.-Naturwiss. Klasse*, No. 5, 275, 281 (1950); *Zeits. f. Naturf.*, **7a**, 66 (1952); F. CAP: *Endliche Feldenergie und Form der Lagrangefunktion* (in preparation); R. FINKELSTEIN: *Phys. Rev.*, **75**, 1079 (1949).

⁽¹²⁾ E. LÜDERS *et al.*: *Zeits. f. Naturf.*, **7a**, 213 (1952); R. MOORHOUSE: *Adv. in Phys.*, **2**, 185 (1953); W. HEISENBERG: *Festschrift 200 Jahre Akad. Göttingen*, p. 50 (1951).

⁽¹³⁾ L. SCHIFF: *Phys. Rev.*, **84**, 1 (1951); **84**, 10 (1951); K. WILDERMUTH *et al.*: *Zeits. f. Naturf.*, **9a**, 799 (1954); P. URBAN and K. WILDERMUTH: *Zeits. f. Naturf.*, **9a**, 748 (1954); W. THIRRING: *Helv. Phys. Acta*, **26**, 33 (1952); *Zeits. f. Naturf.*, **7a**, 63 (1951); FERNBACH *et al.*: *Phys. Rev.*, **75**, 1352 (transparency and non-linear terms).

which is postulated also for other reasons ⁽¹⁴⁾; also a $\lambda\varphi^4$ -term seems to be necessary ⁽¹⁵⁾.

7) Meson-meson scattering ⁽¹⁵⁾ and the decay of the τ -meson into three pions ⁽¹⁶⁾ seem to indicate that between pions we have non-electric forces so that the pion field equation has to be non-linear.

8) The Foldy transformation of the PSpv theory yields non-linear terms ⁽¹⁷⁾.

9) The proof of the production of more than one or two pions in one single nucleon-nucleon collision ⁽¹⁸⁾.

10) Indication of non-linear effects at the computation of the magnetic nucleon moment in the PS theory ⁽¹⁹⁾.

These conclusions, however, are valid only under the condition that not non-linear effects can be produced already by linear theories ⁽²⁰⁾. For the computation of the deuteron a potential $R(x)$ regular at zero (see Fig. 1) was used which resulted from a non-linear PS meson theory devised by F. CAP ⁽²¹⁾. The case II — named « Dipole solution » in one of these works (F. CAP, 1955) — leads (see cit.) to the following static interaction $V_1(x)$ between two nucleons (see Fig. 2)

$$(3) \quad V_1(x) = g^2 \kappa (\boldsymbol{\tau}_1 \boldsymbol{\tau}_2) \left\{ (\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2) \frac{R}{x} + \frac{1}{3} (\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2) x \frac{d}{dx} \left(\frac{R}{x} \right) + S_{12} x \frac{d}{dx} \left(\frac{R}{x} \right) \right\} = \\ = V(x) + S_{12} \bar{V}(x),$$

with $\boldsymbol{\tau}^2 = 1$ and $\kappa r = x$, r [cm], $\kappa = 7.149 \cdot 10^{12} \text{ cm}^{-1}$ (from pion mass).

⁽¹⁴⁾ MALENKA: *Phys. Rev.*, **85**, 686 (1952); MATTHEWS: *Phil. Mag.*, **41**, 185 (1950); GLAUBER: *Phys. Rev.*, **85**, 395 (1951); GEILIKMAN: *Dokl. Akad. Nauk SSSR*, **91**, 225 (1953); IVANENKO: *Dokl. Akad. Nauk SSSR*, **88**, 425 (1953).

⁽¹⁵⁾ SALAM: *Phys. Rev.*, **82**, 217 (1951) (Meson-Meson-Scattering); W. THIRRING: *Zeits. f. Naturf.*, **7a**, 63 (1951); K. BRUECKNER and K. WATSON: *Phys. Rev.*, **87**, 621 (1952) (Meson-Meson-Interaction); D. IVANENKO and V. LEBEDEV: *Žurnal eksper. teor. fiz.*, **22**, 638 (1952).

⁽¹⁶⁾ BERGMANN: *Phys. Rev.*, **95**, 174 (1954); BERESTECKIJ: *Dokl. Akad. Nauk SSSR*, **92**, 519 (1953).

⁽¹⁷⁾ L. SCHIFF: *Phys. Rev.*, **84**, 10 (1951); J. STEINBERGER *et al.*: *Phys. Rev.*, **84**, 581 (1951); K. BRUECKNER and K. WATSON: *Phys. Rev.*, **90**, 699 (1953).

⁽¹⁸⁾ W. HEISENBERG: *Kosmische Strahlung* (Berlin, 1953); *Zeits. f. Phys.*, **126**, 569 (1949); D. IVANENKO *et al.*: *Dokl. Akad. Nauk SSSR*, **84**, 683 (1952); **80**, 357 (1951); **78**, 889 (1951); **88**, 425 (1953); J. J. LORD, J. FAINBERG *et al.*: *Nuovo Cimento*, **7**, 774 (1950); *Nature*, **163**, 47 (1949).

⁽¹⁹⁾ L. SCHIFF: *Phys. Rev.*, **84**, 1, 10, 395 (1952); MALENKA: *Phys. Rev.*, **85**, 686 (1952).

⁽²⁰⁾ J. RAYSKI: *Acta Phys. Pol.*, **10**, 151, 207 (1951).

⁽²¹⁾ a) F. CAP: *Progr. Theor. Phys.*, **10**, 235 (1953); b) *Phys. Rev.*, **95**, 287 (1954); c) *Progr. Theor. Phys.* (in press).

At zero the interaction energy is singular only as $1/x$; the energy density belonging to the potential is everywhere finite ^(21c). The only free parameter is the coupling constant g ; its magnitude, however, is fixed by the rest energy of the nucleon ^(21c).

The usual method ⁽⁶⁾, namely to assume at the integration of the deuteron equations for $r < r_c$ the wave function as given and to integrate the deuteron

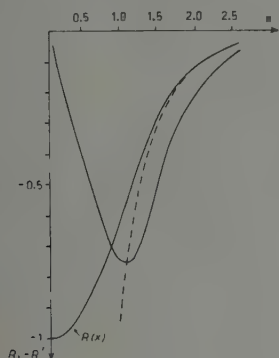


Fig. 1. — Dipole Potential Function $R(x)$ dotted: Derivative of e^{-x}/x .

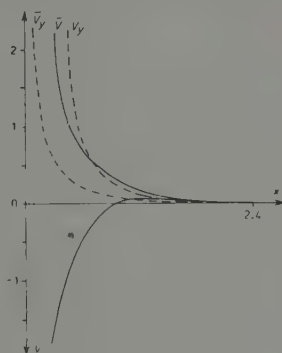


Fig. 2. — The Interaction Energies V, \bar{V} ; dotted: Yukawa Function V, \bar{V} .

equations only from $r > r_c$ on, fails here: the integration has to start at $r=0$.

As it is known, the differential equations for the wave functions of the deuteron (triplet)

$${}^3S \text{ wave } \frac{u}{r} \quad {}^3D \text{ wave } \frac{v}{r\sqrt{8}}$$

read as follows ⁽²²⁾

$$(4) \quad \begin{cases} u'' - \frac{1}{L^2} \left(1 + \frac{1}{\varepsilon} V \right) u - \frac{\sqrt{8}}{\varepsilon L^2} \bar{V} v = 0 \\ v'' - \frac{\sqrt{8}}{\varepsilon L^2} \bar{V} u - \frac{1}{L^2} \left(1 + \frac{6L^2}{r^2} + \frac{1}{\varepsilon} V - \frac{2}{\varepsilon} \bar{V} \right) v = 0 \end{cases}$$

where u and v are at least twice continuously differentiable functions of r in the interval $0 \leq r \leq \infty$ which satisfy the boundary conditions

$$(5) \quad u(0) = v(0) = 0, \quad \lim_{r \rightarrow \infty} u(r) = \lim_{r \rightarrow \infty} v(r) = 0$$

⁽²²⁾ G. EDER: *Acta Phys. Austr.*, **8**, 245 (1952); RARITA and SCHWINGER: *Phys. Rev.*, **59**, 436 (1941); BROyles and HU: *Phys. Rev.*, **79**, 247 (1950).

and the normalization condition

$$(6) \quad \int_0^{\infty} (u^2 + v^2) dr = 1.$$

We furthermore have

$$(7) \quad \begin{cases} \varepsilon = 3,5676 \cdot 10^{-6} \text{ [erg]} = 2,227 \text{ [MeV]} \\ \frac{1}{L^2} = \frac{M\varepsilon}{\hbar^2} \quad L = 4,314 \cdot 10^{-13} \text{ [cm]}. \end{cases}$$

In order that (4) has a solution exactly for the experimentally measured binding energy ε , g has to have a quite definite value. This value is considered by us as the eigenvalue to be found. A first indication for g is obtained from the Ritz method. The unitary theory yields ^(21c) from the rest energy of the nucleon:

$$(8) \quad g \sim 4.89 \cdot 10^{-9} \text{ [cm}^{\frac{3}{2}} \text{ s}^{-1}\text{]}.$$

2. - Approximative solution according to Ritz.

The variational principle belonging to (4) was set up and the following trial functions were tried out:

$$a) \quad \begin{cases} u = ar \exp[-\kappa r], & V = \kappa^2 \beta \left\{ \frac{R}{x} + \frac{1}{3} x \frac{d}{dx} \left(\frac{R}{x} \right) \right\} \\ v = br \exp[-\kappa r], & \bar{V} = \kappa^2 \beta \frac{1}{3} x \frac{d}{dx} \left(\frac{R}{x} \right) \end{cases} \quad \beta = -\frac{3g^2}{\kappa}.$$

We obtained

$$(9) \quad g = 1.813 \cdot 10^{-9} \text{ [cm}^{\frac{3}{2}} \text{ s}^{-1}\text{]}, \quad Q = 2.882 \cdot 10^{-27} \text{ cm}^2,$$

$$b) \quad \begin{cases} u = ar^2 \exp[-\kappa r] \\ v = br^3 \exp[-\kappa r], \end{cases} \quad V, \bar{V}, \beta \text{ like under } a).$$

These trial functions were chosen in such a way that (4) was well satisfied in the neighbourhood of $r = 0$. We obtained

$$(10) \quad g = 2.2434 \cdot 10^{-9} \text{ [cm}^{\frac{3}{2}} \text{ s}^{-1}\text{]}, \quad Q = 7,0320 \cdot 10^{-27} \text{ cm}^2.$$

This shows that the results of the Ritz method depend very strongly from the choice of the trial functions, and that the computation of the quadrupole moment Q is a very sensitive criterion ⁽²³⁾. In order to obtain more reliable values, different attempts for the numerical integration of (4) were made, partly with g being assumed as given, partly with the intention to compute g directly. For different reasons, all these attempts failed ⁽²⁴⁾. Finally, the following method (W. GRÖBNER) proved to be successful:

3. - The new integration method.

After a transformation of variables $x = \kappa r$ we have with

$$\beta_1 = \frac{\beta}{\varepsilon L^2}, \quad \alpha = \frac{1}{Lk} = 0,32425 \quad \text{from (4)}$$

$$(11) \quad \begin{cases} \frac{d^2 u}{dx^2} - \alpha^2 u = \beta_1 (A_1 u + B_1 v) \\ \frac{d^2 v}{dx^2} - \left(\alpha^2 + \frac{6}{x^2} \right) v = \beta_1 (B_1 u + C_1 v), \end{cases}$$

with

$$A_1 = \frac{V}{\beta \kappa^2}, \quad B_1 = \frac{\sqrt{8} \bar{V}}{\beta \kappa^2}, \quad C_1 = \frac{V - 2\bar{V}}{\beta \kappa^2},$$

being tabulated functions of x ; the definition of $R(x)$ is to be found in ^(21c).

The solution of (5), (6), (11) shall be obtained by an iteration procedure, i.e. we write (11) in the following way:

$$(12) \quad \begin{cases} \frac{d^2 u_{v+1}}{dx^2} - \alpha^2 u_{v+1} = \beta_1 (A_1 u_v + B_1 v_v) \\ \frac{d^2 v_{v+1}}{dx^2} - \left(\alpha^2 + \frac{6}{x^2} \right) v_{v+1} = \beta_1 (B_1 u_v + C_1 v_v) \end{cases} \quad v = 0, 1, 2, \dots$$

Then we start with an arbitrarily chosen pair of functions u_0, v_0 that has to satisfy only the boundary conditions (5) and the condition (6); we substitute it into the right member of (12) and thus compute u_1, v_1 under consideration of the boundary conditions (5) and the normalization (6). The latter

⁽²³⁾ W. HEISENBERG: *Progr. Theor. Phys.*, **5**, 523 (1950); TAKETANI: *Progr. Theor. Phys.*, **6**, 581 (1951).

⁽²⁴⁾ See: EOARDC Rep. 633-C.

one yields the value for the still undetermined factor of proportionality β_1 which thus is determined except for the sign.

In order to be able to carry out these computations, we have to solve the ordinary differential equations

$$(13) \quad \begin{cases} u'' - \alpha^2 u = f(x) & \text{with } u(0) = u(\infty) = 0 \\ v'' - \left(\alpha^2 + \frac{6}{x^2}\right) v = g(x) & \text{with } v(0) = v(\infty) = 0. \end{cases}$$

These solutions are known:

$$u = c \sinh \alpha x + \frac{1}{\alpha} \int_0^x \sinh \alpha(x - \xi) f(\xi) d\xi,$$

where the constant c has to be determined satisfying the condition $u(\infty) = 0$, furthermore

$$(15) \quad v = c_1 \exp[\alpha x] p_1(x) + c_2 \exp[-\alpha x] p_2(x) + \frac{1}{2\alpha^2} \int_0^x \{ \exp[\alpha(x - \xi)] p_1(x) \cdot \\ \cdot p_2(\xi) - \exp[-\alpha(x - \xi)] p_1(\xi) p_2(x) \} g(\xi) d\xi,$$

where the constants c_1, c_2 have to be determined satisfying the conditions $v(0) = v(\infty) = 0$.

We furthermore have:

$$p_1(x) = \alpha^3 - \frac{3\alpha^2}{x} + \frac{3\alpha}{x^2}, \quad p_2(x) = \alpha^3 + \frac{3\alpha^2}{x} + \frac{3\alpha}{x^2}.$$

The computation of these integrals, however, would be extremely laborious with a little more complicated functions $f(x)$ and $g(x)$ such as would be obtained with the iteration procedure rather soon. In order to simplify these numerical computations, we used expansions following the Laguerre function systems. The following two systems of Laguerre functions were used:

$$(16) \quad \psi_{1n}(x) = \frac{\sqrt{2\alpha}}{n!} \exp[\alpha x] \frac{d^n}{dx^n} (\exp[-2\alpha x] x^n); \quad n = 1, 0, 2, \dots$$

$$(17) \quad \psi_{3n}(x) = \frac{2\alpha\sqrt{2\alpha}}{n!\sqrt{(n+1)(n+2)}} \exp[\alpha x] \frac{d^n}{dx^n} (\exp[-2\alpha x] x^{n+2}); \quad n = 0, 1, 2, \dots$$

These functions are orthogonal and normal within the interval of $0 \leq x < \infty$:

$$(18) \quad \int_0^{\infty} \psi_{1n}(x) \psi_{1m}(x) dx = \delta_{nm}$$

$$(19) \quad \int_0^{\infty} \psi_{3n}(x) \psi_{3m}(x) dx = \delta_{nm}$$

$$\delta_{nm} = \begin{cases} 0 & \text{for } n \neq m \\ 1 & \text{for } n = m. \end{cases}$$

The first six functions of the system (16) are as follows:

$$\begin{aligned} \psi_{10}(x) &= \sqrt{2\alpha} \exp[-\alpha x], \\ \psi_{11}(x) &= \sqrt{2\alpha} \exp[-\alpha x] (1 - 2\alpha x), \\ \psi_{12}(x) &= \sqrt{2\alpha} \exp[-\alpha x] (1 - 4\alpha x + 2\alpha^2 x^2), \\ \psi_{13}(x) &= \sqrt{2\alpha} \exp[-\alpha x] \left(1 - 6\alpha x + 6\alpha^2 x^2 - \frac{4}{3} \alpha^3 x^3\right), \\ \psi_{14}(x) &= \sqrt{2\alpha} \exp[-\alpha x] \left(1 - 8\alpha x + 12\alpha^2 x^2 - \frac{16}{3} \alpha^3 x^3 + \frac{2}{3} \alpha^4 x^4\right), \\ \psi_{15}(x) &= \sqrt{2\alpha} \exp[-\alpha x] \left(1 - 10\alpha x + 20\alpha^2 x^2 - \frac{40}{3} \alpha^3 x^3 + \frac{10}{3} \alpha^4 x^4 - \frac{4}{15} \alpha^5 x^5\right). \end{aligned}$$

The first six functions of the system (19) are as follows:

$$\begin{aligned} \psi_{30}(x) &= \sqrt{\alpha} \exp[-\alpha x] (2\alpha x), \\ \psi_{31}(x) &= 2 \sqrt{\frac{\alpha}{3}} \exp[-\alpha x] (3\alpha x - 2\alpha^2 x^2), \\ \psi_{32}(x) &= 2 \sqrt{\frac{2\alpha}{3}} \exp[-\alpha x] (3\alpha x - 4\alpha^2 x^2 + \alpha^3 x^3), \\ \psi_{33}(x) &= 2 \sqrt{10\alpha} \exp[-\alpha x] \left(\alpha x - 2\alpha^2 x^2 + \alpha^3 x^3 - \frac{2}{15} \alpha^4 x^4\right), \\ \psi_{34}(x) &= \sqrt{\frac{5\alpha}{3}} \exp[-\alpha x] \left(6\alpha x - 16\alpha^2 x^2 + 12\alpha^3 x^3 - \frac{16}{5} \alpha^4 x^4 + \frac{4}{15} \alpha^5 x^5\right), \\ \psi_{35}(x) &= 2 \sqrt{\frac{7\alpha}{3}} \exp[-\alpha x] \left(3\alpha x - 10\alpha^2 x^2 + 10\alpha^3 x^3 - 4\alpha^4 x^4 + \frac{2}{3} \alpha^5 x^5 - \frac{4}{105} \alpha^6 x^6\right). \end{aligned}$$

The system of the functions $\psi_{1n}(x)$ is suited for the expansion of the functions $f(x)$ and $g(x)$ according to the formulas:

$$(20) \quad f(x) = \sum_{n=0}^{\infty} c_n \psi_{1n}(x); \quad g(x) = \sum_{n=0}^{\infty} d_n \psi_{1n}(x)$$

with the coefficients of expansion:

$$(21) \quad c_n = \int_0^\infty f(x) \psi_{1n}(x) dx, \quad d_n = \int_0^\infty g(x) \psi_{1n}(x) dx; \quad n = 0, 1, 2, \dots$$

It is only natural in numerical computations to be satisfied with a finite intercept of the infinite series (20). The system of the functions $\psi_{3n}(x)$ which satisfies the boundary conditions therefore is suited for the expansion of the solutions of the differential equations. Once the functions $f(x)$ and $g(x)$ have been expanded it is possible to write down immediately the solutions of these differential equations expanded according to the orthogonal system of functions $\psi_{3n}(x)$, if the following charts are used:

Solutions of $u'' - \alpha^2 u = \psi_{1n}$.

$$\begin{aligned} n = 0: \quad u &= -\frac{\sqrt{2}}{4\alpha^2} \psi_{30}, \\ n = 1: \quad u &= \frac{1}{8\alpha^2} (3\sqrt{2} \psi_{30} - \sqrt{6} \psi_{31}), \\ n = 2: \quad u &= \frac{1}{24\alpha^2} (-3\sqrt{2} \psi_{30} + 5\sqrt{6} \psi_{31} - 4\sqrt{3} \psi_{32}), \\ n = 3: \quad u &= \frac{1}{24\alpha^2} (-2\sqrt{6} \psi_{31} + 7\sqrt{3} \psi_{32} - 3\sqrt{5} \psi_{33}), \\ n = 4: \quad u &= \frac{1}{40\alpha^2} (-5\sqrt{3} \psi_{32} + 9\sqrt{5} \psi_{33} - 2\sqrt{30} \psi_{34}), \\ n = 5: \quad u &= \frac{1}{120\alpha^2} (-12\sqrt{5} \psi_{33} + 11\sqrt{30} \psi_{34} - 5\sqrt{42} \psi_{35}); \end{aligned}$$

Solutions of $v'' - (\alpha^2 + 6/x^2)v = \psi_{1n}$.

$$\begin{aligned} n = 0: \quad v &= -\alpha \sqrt{2\alpha} v_a - \frac{3\sqrt{2}}{16\alpha^2} \psi_{30} + \frac{\sqrt{6}}{16\alpha^2} \psi_{31}, \\ n = 1: \quad v &= -3\alpha \sqrt{2\alpha} v_a - \frac{3\sqrt{2}}{16\alpha^2} \psi_{30} + \frac{\sqrt{6}}{16\alpha^2} \psi_{31}, \\ n = 2: \quad v &= -5\alpha \sqrt{2\alpha} v_a - \frac{11\sqrt{2}}{16\alpha^2} \psi_{30} + \frac{19\sqrt{6}}{48\alpha^2} \psi_{31} - \frac{1}{2\sqrt{3}\alpha^2} \psi_{32}, \\ n = 3: \quad v &= -7\alpha \sqrt{2\alpha} v_a - \frac{13\sqrt{2}}{16\alpha^2} \psi_{30} + \frac{13\sqrt{6}}{48\alpha^2} \psi_{31} + \frac{5}{8\sqrt{3}\alpha^2} \psi_{32} - \frac{\sqrt{5}}{8\alpha^2} \psi_{33}, \\ n = 4: \quad v &= -9\alpha \sqrt{2\alpha} v_a - \frac{21\sqrt{2}}{20\alpha^2} \psi_{30} + \frac{9\sqrt{6}}{20\alpha^2} \psi_{31} - \frac{13\sqrt{3}}{80\alpha^2} \psi_{32} + \frac{3\sqrt{5}}{16\alpha^2} \psi_{33} - \frac{\sqrt{30}}{20\alpha^2} \psi_{34}. \end{aligned}$$

The function v_a given here is the solution of the differential equation $v'' - (\alpha^2 + 6/x^2)v = x \exp[-\alpha x]$, which satisfies the boundary conditions (5).

That we have $v_a(0) = 0$ is indicated by an expansion which is valid in the neighbourhood of $x = 0$.

Once we have expanded $f(x)$ and $g(x)$ according to (20) following the orthogonal system $\psi_{1n}(x)$, then it is possible with the help of the charts to write down the solutions u and v as expanded according to the orthogonal system $\psi_{3n}(x)$.

The iteration method converges in analogy to the Bernoulli method for solution of algebraic equations, and this under the condition that the eigenvalue problem has a discrete spectrum of eigenvalues and eigenfunctions belonging hereto which form a complete orthogonal system, and that the numerically smallest eigenvalue is simple. Then it converges towards the solution belonging to this eigenvalue. This convergence is the better the larger the absolute difference between the smallest eigenvalue and the other eigenvalues. Because in the deuteron problem we have to expect only one eigenvalue we may regard the good convergence of the method as likely.

Unfortunately, lack of time did not permit us to continue the numerical computations to such a stage as would have been desirable, nevertheless, the approximation obtained in three steps should be satisfactory

$$u_0(x) = v_0(x) = \frac{1}{\sqrt{2}} \psi_{10}(x),$$

was chosen as starting point for the first step. Here the normalisation (5) was substituted by the following one:

$$(22) \quad \int_0^\infty (u^2 + v^2) dx = 1,$$

therefore all computed functions $u(x)$, $v(x)$ have to be multiplied by $\sqrt{\pi} = 2.6738 \cdot 10^6$.

After the third step we received $g = 1.8180 \cdot 10^{-9}$

$$u_3(x) = -0.8220\psi_{30} - 0.3965\psi_{31} - 0.2289\psi_{32} - 0.1172\psi_{33} - 0.1729\psi_{34},$$

$$v_3(x) = 2.2546v_a + 13.6903\psi_{30} - 9.9299\psi_{31} + 0.7286\psi_{32} + 0.4131\psi_{33} + 0.1079\psi_{34},$$

and for the quadrupole moment we obtained:

$$(23) \quad Q = 2.80 \cdot 10^{-27} \text{ cm}^2 \quad \text{exper.: } 2.73 \cdot 10^{-27} \text{ cm}^2$$

$$(24) \quad \mu_D = 0.83925 \text{ N.M.} \quad \text{exper.: } 0.85735 \text{ N.M.}$$

$$(25) \quad p_D = 7.11\% \quad \text{exper.: } 4 \div 8\%$$

whilst LÉVY (4) obtained $Q = 2.08 \cdot 10^{-27} \text{ cm}^2$ using the second and fourth order potential of the linear PS theory. It might be of interest, that the long tail of the Yukawa potential is important ⁽²³⁾:

$$Q_1 = \int_0^3 \rightarrow 1,43928 \cdot 10^{-27} \text{ cm}^2,$$

$$Q_2 = \int_3^\infty \rightarrow 1,36539 \cdot 10^{-27} \text{ cm}^2.$$

For $x > 3$ we used the well known asymptotic solutions.

Our result seems to indicate that the study of regular potentials seems to be important.

The research reported in this publication has been made possible through financial support by the US Air Force under contract No. AF 61 (514)-633-C.

The authors want to express their sincere gratitude to the European Office OARDC for the sponsorship of this work. Thanks are also due to MM. KNOPP and GRÖMER for the execution of numerical computations.

RIASSUNTO (*)

Si discutono in primo luogo il successo e il fallimento della teoria mesonica pseudoscalare delle forze nucleari e si dà ragion dei termini non lineari che danno potenziali finiti. Si dà poi un nuovo metodo per la soluzione del problema del deutone e lo si applica a un potenziale pseudoscalare regolare. Si dimostra che la teoria pseudoscalare non lineare discussa fornisce per il deutone risultati assai migliori di quelli forniti dalla teoria lineare.

(*) *Traduzione a cura della Redazione.*

La production des mésons et les états excités à vie moyenne brève.

B. D'ESPAGNAT et J. PRENTKI

CERN - Genève

(ricevuto il 4 Maggio 1955)

Summary. — This paper concerns the models for meson production which make use of an intermediary excited nucleon X having a short lifetime, E_X . $\pi + \mathcal{N} \rightarrow \pi + X \rightarrow \pi + \pi + \mathcal{N}$. The legitimacy of drawing from such a hypothesis, together with charge independence, simple clear cut conclusions as to the relative numbers of mesons created with various charges is discussed. It is shown that, contrary to what might be expected at first sight, these ratios depend, in a complicated way, on the line width (lifetime) of the excited state. Only in the case of a line width small compared to excitation energy do they reach the values given by the more customary and elementary approach. Isotopic spin conservation is assumed. For simplicity calculations have been made using scalar meson theory without recoil; reactions (1) and (2) are treated as an example. The method consists in calculating the matrix elements of the T matrix for the whole process, i.e. without supposing implicitly that X be real. The intermediary state hypothesis is introduced by keeping only the graphs of Fig. 3. The important contributions of damping, given by the graphs of Fig. 2, are taken into account by solving the appropriate Heitler equations. The general result for our example is given by (22). A discussion of this formula shows that the limiting value of σ_{0+}/σ_{++} can hardly be a good approximation unless the line width Γ of the excited state is less than about a tenth of its excitation energy Δ . We conclude that when X is identified with the $I=J=3/2$; $\Delta=150$ MeV, $\Gamma \approx 80$ MeV excited state of the nucleon, the elementary approach for the calculation of such ratios should not be considered as reliable.

1. — Introduction.

Il a été récemment suggéré d'interpréter certains résultats expérimentaux au moyen d'états excités à vie moyenne très brève. Cette hypothèse a été émise à propos de la production des mésons π soit dans les chocs nucléon-

nucléon soit dans les chocs méson π -nucléon. En ce qui concerne les chocs nucléon-nucléon on admet que dans une première étape, il y a excitation d'un des deux nucléons; la deuxième étape consiste en la désexcitation de ce nucléon, avec émission d'un méson π . Pour des énergies suffisamment grandes les deux nucléons peuvent être excités simultanément.

Cette hypothèse est, sous certains aspects, assez attrayante. En effet le modèle statistique de Fermi pour la production multiple des mésons — qui avait été d'abord appliqué sans modifications — fournit un rapport σ_2/σ_1 à peu près vingt fois inférieur au rapport observé par SHUTT *et al.* ⁽¹⁾ pour des énergies de 2 GeV environ (σ_2 =section efficace de production double, σ_1 =celle de production simple). Il est clair que l'hypothèse du passage par des états excités, correspondant à des réactions où interviennent des phénomènes de résonance, est un moyen possible de modifier sensiblement ce rapport théorique en faveur de σ_2 dès que les énergies le permettent. Naturellement il aurait aussi été possible d'augmenter σ_2/σ_1 , en restant dans le cadre de la théorie de Fermi. Il aurait suffi de modifier convenablement le « volume caractéristique ». Toutefois ⁽¹⁾ une telle opération aurait également pour effet d'augmenter le nombre des productions triples et cela bien au dessus du nombre expérimentalement observé. Par ailleurs il semble que dans les chocs $N + P \rightarrow N + P + \pi^+ + \pi^-$ une certaine corrélation ait été observée ⁽¹⁾ entre le proton et le méson π^+ d'une part et le neutron et le méson π^- de l'autre. Ceci pourrait s'expliquer par la formation de deux états excités à condition que ceux ci aient un spin isotopique 3/2.

Enfin YUAN et LINDENBAUM ⁽²⁾ ont étudié la production des mésons π dans les chocs nucléon-nucléon aux énergies de 1 GeV et 2,3 GeV (en prenant pour cible le Be et en observant les mésons créés sous un angle de 80° à 100° par rapport à la direction des nucléons dans le système du centre des masses.) Leurs résultats indiquent fortement l'existence d'un maximum dans la distribution en énergie des mésons, maximum centré autour de 100-150 MeV, ceci aussi bien pour des protons de 1 GeV que de 2,3 GeV. Ces résultats peuvent de nouveau s'interpréter à l'aide de l'hypothèse précédente et de plus suggèrent, vu la similarité des énergies d'excitation, une identification de l'état excité en question avec celui dont les apparences se manifestent dans la diffusion élastique des mésons π par l'hydrogène.

La création des mésons π dans les collisions $\pi + \mathcal{N} \rightarrow \pi + \pi + \mathcal{N}$ a été étudiée expérimentalement par WALKER, CRUSSARD et KOSHIBA ⁽³⁾ et par

⁽¹⁾ W. B. FOWLER, R. P. SHUTT, A. M. THORNDIKE et W. L. WHITTEMORE: *Phys. Rev.*, **95**, 1026 (1954).

⁽²⁾ L. C. L. YUAN et S. J. LINDENBAUM: *Phys. Rev.*, **93**, 1431 (1954); S. J. LINDENBAUM et L. C. L. YUAN: *Phys. Rev.*, **95**, 638 (1954).

⁽³⁾ W. D. WALKER, J. CRUSSARD et M. KOSHIBA: *Phys. Rev.*, **95**, 852 (1954).

SHUTT et collaborateurs (4). En ce qui concerne la manifestation d'un état excité par des considérations énergétiques la situation est assez similaire à celle décrite ci-dessus, bien que moins nette. Des calculs simples, basés sur la conservation du spin isotopique et du type de ceux que nous allons discuter, n'ont pas donné de résultats très concluants.

Du point de vue théorique, si l'idée de deux étapes successives est prise au pied de la lettre, si de plus on admet l'hypothèse de l'invariance du spin isotopique, alors pour une valeur donnée du spin isotopique de l'état excité du nucléon il est possible par un raisonnement simple (voir section 2) de trouver des relations entre les sections efficaces de production de mésons π de différentes charges. Ainsi PEASLEE (5) a appliqué ce modèle à la production d'un et de deux mésons dans les chocs nucléon-nucléon. Il a montré par exemple que pour des collisions proton-proton avec émission d'un méson π le rapport σ_+/σ_0 est égal à 5. De même on obtient dans les chocs $\pi^+ + P$ un rapport σ_{+0}/σ_{++} égal à 13/2, tout ceci en supposant que l'état excité est de spin isotopique $3/2$.

Dans le présent article nous ne voulons pas discuter la plus ou moins grande validité de l'hypothèse selon laquelle la totalité de la réaction s'effectuerait par l'intermédiaire d'un — ou de deux — nucléons excités. Nous voulons seulement nous préoccuper de rechercher si, cette hypothèse étant admise, des conclusions du type précédent — rapports bien définis entre nombres de mésons de différentes charges par exemple — en découlent nécessairement. Notre thèse sera, comme l'un de nous l'a déjà suggéré (B.E.) (6), que si la vie moyenne de l'état excité est raisonnablement longue de telles conclusions sont effectivement valables mais que, par contre, si cette vie moyenne est très courte, bien que l'on se limite à un état excité de spin isotopique donné, le calcul en deux étapes peut être erroné (7). Cette remarque est en particulier valable si le nucléon excité est assimilé, comme indiqué plus haut, à celui qui interviendrait dans les collisions élastiques $\pi^+ + P$ et dont on sait que la vie moyenne serait très brève ($\sim 10^{-23}$ s).

Il pourrait sembler qu'une telle remarque est évidente (bien qu'elle n'ait pas toujours été appliquée!). En effet elle est en rapport manifeste avec la quatrième relation d'incertitude. Toutefois il faut noter que cette relation concerne essentiellement les énergies et que d'habitude, lorsque l'on fait appel à des états excités, on traite des problèmes très différents de celui qui nous occupe (par exemple l'excitation d'atomes dans les collisions ou la théorie du noyau composé) où la limitation due à la relation de Heisenberg est immé-

(4) L. M. EISBERG, W. B. FOWLER, R. M. LEA, W. D. SHEPHARD, K. P. SHUTT, A. M. THORNDIKE et W. L. WHITEMORE: *Phys. Rev.*, **97**, 797 (1955).

(5) D. C. PEASLEE: *Phys. Rev.*, **94**, 1085 (1954) et **95**, 1580 (1954).

(6) B. D'ESPAGNAT: *Compt. Rend.*, **240**, 164 (1955).

(7) B. D'ESPAGNAT et J. PRENTKI: *Compt. Rend.*, **240**, 1514 (1955).

diate. Cependant le spin isotopique est une grandeur sans aucune relation avec la vie moyenne et il ne serait pas exclu, à priori, que des résultats déduits uniquement à l'aide de cette grandeur soient, en fait, indépendants de la vie moyenne. Il n'est pas difficile de trouver des phénomènes où il en va ainsi. Citons, à titre d'exemple simple, la diffusion élastique π -nucléon si on la suppose entièrement due à un état excité intermédiaire. Enfin nous nous efforcerons de donner aussi une idée grossière de l'erreur que l'on commet en admettant une telle hypothèse.

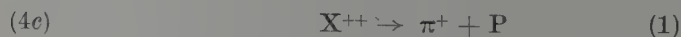
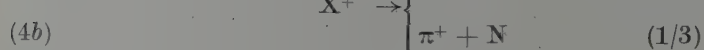
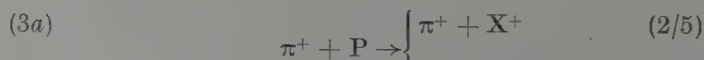
Les processus (1) et (2) sont pris comme exemple. Dans la section 2 un bref rappel de la méthode élémentaire est donné. Il est montré qu'elle est peu satisfaisante pour des vies moyennes brèves et le principe de la méthode que nous utilisons est présenté. La section 3 est consacrée au calcul du rapport entre les sections efficaces de (1) et (2). Enfin dans les section 4 et 5 le comportement de ce rapport est étudié en fonction de la vie moyenne et certaines conclusions en sont dégagées.

2. — Préliminaires.

Nous rappelons brièvement la méthode utilisée en particulier par PEASLEE (⁵). Pour fixer les idées nous l'appliquerons au problème très simple, caractérisé par les deux réactions suivantes:



Les deux étapes du calcul sont ici



(1) ne peut résulter que de la succession (3a), (4b) tandis que (2) est une combinaison de (3b) suivie par (4c) et de (3a) suivie par (4a).

X représente un état excité de spin isotopique 3/2 du nucléon. Les nombres entre parenthèses figurant dans (3ab) et (4abc) sont les probabilités relatives de ces différentes transitions, le spin isotopique total étant conservé.

Le rapport σ_{++}/σ_{+0} est ainsi

$$\left(\frac{2}{5} \cdot \frac{1}{3}\right) / \left(\frac{3}{5} \cdot 1 + \frac{2}{5} \cdot \frac{2}{3}\right) = \frac{2}{13}.$$

Ce calcul, ainsi d'ailleurs que tous les calculs habituels faisant intervenir des états excités, est en somme basé sur la possibilité de traiter au cours de la première étape l'état excité comme un état stable, ce qui est légitime dans le cas des vies moyennes longues. La première étape constitue alors un phénomène en soi auquel s'appliquent les lois de conservation en général et l'état excité est bien entendu réel. Ce n'est que dans la deuxième étape qu'il est tenu compte de l'instabilité de l'état excité. Comme il a été dit dans l'introduction, il existe des cas où les résultats de cette manière de procéder sont en fait plus généraux qu'il ne paraîtrait et corrects, même si la vie moyenne est courte mais ceci, bien entendu, n'est pas le cas général.

Pour savoir si c'est le cas ou non pour le problème qui nous occupe nous devons trouver un moyen de décrire le passage par états excités sans introduire de surcroît l'hypothèse des longues vies moyennes. Reprenant, mais en des termes qui nous semblent plus corrects, la même description physique du phénomène, nous faisons un choix parmi tous les graphes représentant le processus global et nous ne conservons que ceux qui sont représentés par la fig. 1.

L'état initial, composé d'un proton et d'un méson passe vers l'état intermédiaire $X + \pi$ où le méson π est déjà dans son état final. Ceci correspond à la première étape de la description précédente. Le nucléon excité se désintègre ensuite en nucléon plus méson π . Les processus d'ordre quelconque qui sont dissimulés par les cercles n'ont pas à être précisés.

Le calcul qui suit est développé à partir de la fig. 1 selon les méthodes habituelles, tenant compte du damping qui, comme on le verra, joue un rôle important.

Etant donné qu'il s'agit surtout ici d'une question de principe nous discuterons, à titre illustratif, le cas académique d'un méson π scalaire avec couplage scalaire et d'un nucléon infiniment lourd.

3. — Calcul des sections efficaces.

La section efficace de production d'un méson positif d'énergie comprise entre ε_1 et $\varepsilon_1 + d\varepsilon_1$ et d'un autre méson positif d'énergie $\varepsilon_2 = \varepsilon_i - \varepsilon_1$, où ε_i est

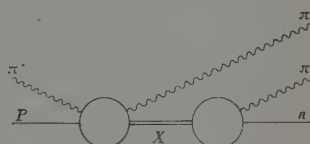


Fig. 1.

l'énergie du méson incident, est donné par (+):

$$(5) \quad \sigma'_{++}(\varepsilon_1) d\varepsilon_1 = (4\pi)^2 2\pi \frac{\varepsilon_i}{k_i} |T_{++}(\varepsilon_1)|^2 \varrho'(\varepsilon_1) d\varepsilon_1$$

avec

$$(5a) \quad k = \sqrt{\varepsilon^2 - \mu^2}; \quad \varrho'(\varepsilon_1) = \frac{k_1 \varepsilon_1 k_2 \varepsilon_2}{(2\pi)^3},$$

et où $T_{++}(\varepsilon_1)$ est l'élément de matrice de la matrice T définie par LIPPMAN et SCHWINGER (8) et qui dans nos hypothèses est indépendante des angles. De même, pour la création d'un méson chargé et d'un méson neutre d'énergies ε_1 et ε_2 respectivement

$$(6) \quad \sigma'_{+0}(\varepsilon_1) d\varepsilon_1 = (4\pi)^2 2\pi \frac{\varepsilon_i}{k_i} |T_{+0}(\varepsilon_1)|^2 \varrho'(\varepsilon_1) d\varepsilon_1.$$

$T_{+0}(\varepsilon_1)$ représenterait dans ces notations l'élément de matrice de la création d'un méson neutre d'énergie ε_1 et d'un chargé d'énergie ε_2 .

Les sections efficaces totales de création de deux mésons à partir d'un méson incident d'énergie ε_i sont:

$$(7) \quad \sigma_{++}(\varepsilon_i) = \frac{1}{2} \int \sigma'_{++}(\varepsilon_1) d\varepsilon_1,$$

$$(8) \quad \sigma_{+0}(\varepsilon_i) = \int \sigma'_{+0}(\varepsilon_1) d\varepsilon_1.$$

Le facteur $\frac{1}{2}$ dans (7) provient du fait que l'on a dans l'état final deux particules identiques.

L'hypothèse de l'invariance du spin isotopique permet de décomposer T_{++} et T_{+0} suivant les valeurs du spin isotopique total et du spin isotopique du groupe des deux mésons finaux (*).

Après un calcul élémentaire, utilisant les propriétés bien connues des coef-

(+) On a posé $\hbar = 1$; $c = 1$.

(8) B. A. LIPPMAN et J. SCHWINGER: *Phys. Rev.*, **79**, 469 (1950).

(*) Une autre possibilité de décomposition serait de grouper un des mésons finaux avec le nucléon ce qui pourrait même sembler plus conforme à la nature du présent problème. La méthode que nous avons choisie s'avère cependant plus maniable.

Je profite de cette note pour signaler que le raisonnement fait en (6) est trop sommaire à cet égard (B.E.).

icients de Clebsch-Gordan on obtient:

$$(9a) \quad T_{++}(\varepsilon_1) = \sqrt{\frac{4}{5}} T^{2\frac{1}{2}}(\varepsilon_1)$$

$$(9b) \quad T_{0+}(\varepsilon_1) = -\sqrt{\frac{1}{5}} \sqrt{\frac{1}{2}} T^{2\frac{1}{2}}(\varepsilon_1) - \sqrt{\frac{1}{2}} T^{1\frac{1}{2}}(\varepsilon_1)$$

$$(9c) \quad T_{+0}(\varepsilon_1) = -\sqrt{\frac{1}{5}} \sqrt{\frac{1}{2}} T^{2\frac{1}{2}}(\varepsilon_1) + \sqrt{\frac{1}{2}} T^{1\frac{1}{2}}(\varepsilon_1)$$

où $T^{I\frac{1}{2}}$ ($I=1$ ou 2) est obtenu en combinant le méson d'énergie ε_1 avec celui d'énergie ε_2 , dans cet ordre, pour obtenir un système de spin isotopique I et ensuite l'ensemble avec le nucléon. On a les propriétés de symétrie suivantes:

$$(10) \quad \begin{cases} T^{2\frac{1}{2}}(\varepsilon_2) = T^{2\frac{1}{2}}(\varepsilon_1) \\ T^{1\frac{1}{2}}(\varepsilon_2) = -T^{1\frac{1}{2}}(\varepsilon_1) \end{cases}$$

ce qui est compatible avec (9bc).

La matrice T se calcule à partir de la matrice de réaction K à l'aide de l'équation de Heitler

$$(11) \quad T_{ba} = K_{ba} - i\pi \sum_c K_{bc} Q_c T_{ca},$$

où c représente tous les états sur le niveau d'énergie $E_a = E_b$.

Limitons nous d'abord à des énergies incidentes inférieures au seuil d'émergence de trois mésons dans l'état final. L'équation (11) donne alors trois équations pour déterminer $T^{I\frac{1}{2}}$ ($I=1, 2$) et $T^{\frac{1}{2}}$ correspondant à la diffusion élastique d'un méson. Dans ce système d'équations interviennent les grandeurs $K^{I\frac{1}{2}}$ et $K^{\frac{1}{2}}$ et aussi les quantités $K^{II'\frac{1}{2}}$ décrivant les transitions de deux mésons à deux mésons (I, I' , sont les spins isotopiques des deux mésons dans l'état initial et final respectivement).

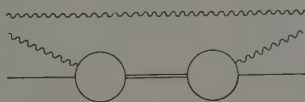


Fig. 2.

Un raisonnement direct sur les éléments de matrice de la matrice K — qui aurait dispensé de résoudre les équations de Heitler — n'aurait pas préservé l'unitarité de la matrice S et la présence de résonances dans le problème aurait conduit à des expressions du type ∞/∞ . Pour palier à ces inconvénients la résolution des équations de Heitler et, en particulier, l'introduction des grandeurs $K^{II'\frac{1}{2}}$ est essentielle. Pour l'étude semiquantitative que nous nous proposons de faire il suffit de considérer les $K^{II'\frac{1}{2}}$ correspondant aux graphes de la fig. 2.

L'équation de Heitler s'écrit alors:

$$(12) \quad T^{\frac{3}{2}} = K^{\frac{3}{2}} - i\pi (4\pi) \varrho K^{\frac{3}{2}} T^{\frac{3}{2}} - i\pi (4\pi)^2 \sum_i \int \varrho'(\varepsilon_1) K^{\frac{3}{2}}(\varepsilon_1) T^{\frac{3}{2}}(\varepsilon_1) d\varepsilon_1,$$

$$(13) \quad T^{I\frac{3}{2}} = K^{I\frac{3}{2}} - i\pi (4\pi) \varrho K^{I\frac{3}{2}} T^{\frac{3}{2}} - \\ - i\pi (4\pi) \sum_{I'} \{ K^{II'\frac{3}{2}}(\varepsilon_1) \varrho(\varepsilon_1) + (-1)^{I+I'} K^{II'\frac{3}{2}}(\varepsilon_2) \varrho(\varepsilon_2) \} T^{I'\frac{3}{2}}(\varepsilon_1)$$

où toutes les quantités T , K et ϱ' sont fonction de ε_i

$$\varrho(\varepsilon) = \frac{k\varepsilon}{(2\pi)^3}, \quad \varrho = \varrho(\varepsilon_i).$$

Les notations abrégées que nous utilisons demandent un mot d'explication. Les $K^{II'\frac{3}{2}}(\varepsilon_1)$ et $K^{II'\frac{3}{2}}(\varepsilon_2)$ que nous introduisons doivent s'écrire $K^{II'\frac{3}{2}}(\varepsilon_1, \varepsilon_2; \varepsilon_1, \varepsilon_2)$ et $K^{II'\frac{3}{2}}(\varepsilon_2, \varepsilon_1; \varepsilon_2, \varepsilon_1)$ respectivement. Par rapport aux permutations dans les états initial et final ils sont de parités $(-)^{I'}$ et $(-)^{I'}$ respectivement. Tenant compte de cette propriété et des relations (10) on obtient (13), car

$$K^{II'\frac{3}{2}}(\varepsilon_1, \varepsilon_2; \varepsilon_1, \varepsilon_2) \varrho(\varepsilon_1) T^{I'\frac{3}{2}}(\varepsilon_1, \varepsilon_2) + K^{II'\frac{3}{2}}(\varepsilon_1, \varepsilon_2; \varepsilon_2, \varepsilon_1) \varrho(\varepsilon_2) T^{I'\frac{3}{2}}(\varepsilon_2, \varepsilon_1) = \\ = K^{II'\frac{3}{2}}(\varepsilon_1) \varrho(\varepsilon_1) T^{I'\frac{3}{2}}(\varepsilon_1) + (-)^{I'} \cdot (-)^{I'} K^{II'\frac{3}{2}}(\varepsilon_2, \varepsilon_1; \varepsilon_2, \varepsilon_1) \varrho(\varepsilon_2) T^{I'\frac{3}{2}}(\varepsilon_1, \varepsilon_2) = \\ = \{ K^{II'\frac{3}{2}}(\varepsilon_1) \varrho(\varepsilon_1) + (-)^{I'+I} K^{II'\frac{3}{2}}(\varepsilon_2) \varrho(\varepsilon_2) \} T^{I'\frac{3}{2}}(\varepsilon_1).$$

Les fonctions K_{++} et K_{+0} sont, d'après l'hypothèse de la section 2, données par les deux contributions des graphes de la fig. 3a et les deux contributions de la fig. 3b respectivement.

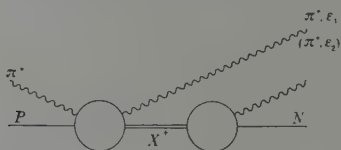


Fig. 3a.

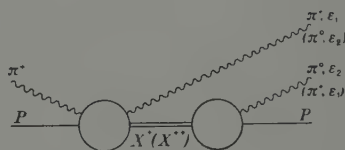


Fig. 3b.

On obtient ainsi, Δ étant l'énergie d'excitation:

$$(14) \quad \left\{ \begin{array}{l} K_{++}(\varepsilon_1) = -\sqrt{\frac{2}{5}} \sqrt{\frac{1}{3}} \left[A(\varepsilon_1) \frac{g_r}{\varepsilon_2 - \Delta} + A(\varepsilon_2) \frac{g_r}{\varepsilon_1 - \Delta} \right] \\ K_{+0}(\varepsilon_1) = -\sqrt{\frac{2}{5}} \sqrt{\frac{2}{3}} A(\varepsilon_1) \frac{g_r}{\varepsilon_2 - \Delta} + \sqrt{\frac{3}{5}} A(\varepsilon_2) \frac{g_r}{\varepsilon_1 - \Delta} \end{array} \right.$$

où $A(\varepsilon_2)$ est une fonction qui n'a pas à être précisée et g_r est la constante de couplage $X\pi\pi$ convenablement renormalisée.

Si l'on pose

$$(14a) \quad Z_{12} = \sqrt{\frac{2}{5}} \sqrt{\frac{1}{3}} A(\varepsilon_1) \frac{g_r}{\varepsilon_2 - \Delta} = Z_{12}^s + Z_{12}^A$$

où Z_{12}^s et Z_{12}^A sont les parties symétrique et antisymétrique de Z_{12} on a à l'aide de (14) et (14a)

$$(15) \quad \begin{cases} K_{++}(\varepsilon_1) = -2Z_{12}^s \\ K_{+0}(\varepsilon_1) = \frac{1}{\sqrt{2}} Z_{12}^s - \frac{5}{\sqrt{2}} Z_{12}^A. \end{cases}$$

Les relations (9) écrites pour les T sont valables pour les K et avec (15) fournissent:

$$(16) \quad \begin{cases} K^{2\frac{1}{2}}(\varepsilon_1) = -\sqrt{5} Z_{12}^s, \\ K^{1\frac{1}{2}}(\varepsilon_1) = -5 Z_{12}^A. \end{cases}$$

Appliquant aux graphes de la fig. 2 une méthode entièrement similaire on obtient:

$$(17) \quad K^{2\frac{1}{2}}(\varepsilon) = -\frac{1}{\sqrt{5}} K^{12\frac{1}{2}}(\varepsilon) = -\frac{1}{\sqrt{5}} K^{21\frac{1}{2}}(\varepsilon) = \frac{1}{5} K^{11\frac{1}{2}}(\varepsilon) = W(\varepsilon).$$

On posera

$$(17a) \quad \varrho(\varepsilon_j) W(\varepsilon_j) = W_j \quad (j = 1, 2),$$

substituant (17a), (17) et (16) dans (12) et (13) on résout les équations de Heitler et on est conduit à:

$$(18a) \quad T^{1\frac{1}{2}}(\varepsilon_1) = -\frac{5}{2} \frac{Z_{12}(1 + i\lambda W_1) - Z_{21}(1 + i\lambda W_2)}{D} \alpha(\varepsilon_i)$$

$$(18b) \quad T^{2\frac{1}{2}}(\varepsilon_1) = -\frac{\sqrt{5}}{2} \frac{Z_{12}(1 + 5i\lambda W_1) + Z_{21}(1 + 5i\lambda W_2)}{D} \alpha(\varepsilon_i)$$

avec

$$(19) \quad \lambda = 8\pi^2; \quad D = 1 + 3i\lambda(W_1 + W_2) - 5\lambda^2 W_1 W_2$$

et où

$$\alpha(\varepsilon_i) = 1 - i \frac{\lambda}{2} \varrho(\varepsilon_i) T^{\frac{1}{2}}$$

est une grandeur qui ne dépend que de ε_i et ne joue aucun rôle dans nos considérations.

Définissons les grandeurs:

$$(20) \quad Y_{12} = Z_{12}(1 + 3i\lambda W_1) + 2i\lambda Z_{21}W_2.$$

Tenant compte de (18a), (18b), (19) (9abc) et (20) on obtient:

$$(21) \quad \begin{cases} T_{++}(\varepsilon_1) = -\frac{Y_{12} + Y_{21}}{D} \alpha(\varepsilon_i) \\ T_{+0}(\varepsilon_1) = \frac{1}{\sqrt{2}} \frac{3Y_{12} - 2Y_{21}}{D} \alpha(\varepsilon_i) \end{cases}$$

d'où facilement à l'aide de (5), (5a), (7), et (8)

$$(22) \quad \frac{\sigma_{+0}}{\sigma_{++}} = \frac{\int |T_{+0}(\varepsilon_1)|^2 \varrho'(\varepsilon_1) d\varepsilon_1}{\frac{1}{2} \int |T_{++}(\varepsilon_1)|^2 \varrho'(\varepsilon_1) d\varepsilon_1} = \frac{(13/2) - 6y}{1 + y}$$

avec

$$(23) \quad y = \frac{\int a(\varepsilon_1) \varrho'(\varepsilon_1) d\varepsilon_1}{\int b(\varepsilon_1) \varrho'(\varepsilon_1) d\varepsilon_1}$$

où

$$(23a) \quad a = \operatorname{Re} \frac{Y_{12}^* Y_{21}}{|D|^2};$$

$$(23b) \quad b = \frac{|Y_{12}|^2}{|D|^2}.$$

Il reste à discuter les valeurs de y données par (23) et (23ab) qui portées dans (22) donnent le rapport que nous étudions. Il est clair que $y = 0$ correspond au résultat donné par la méthode élémentaire dont nous discutons la validité.

4. - Discussion.

D'après les figs. 3 et 2 les formes générales de Z_{12} et W_1 doivent être

$$(24) \quad Z_{12} = \frac{f(\varepsilon_1)}{\varepsilon_2 - \Delta}$$

$$(25) \quad W_1 = \frac{Q}{\lambda} \frac{\varphi(\varepsilon_1)}{\varepsilon_1 - \Delta}$$

où les fonctions $f(\varepsilon_1)$ et $\varphi(\varepsilon_1)$ sont des fonctions régulières de la variable ε_1 , avec

$$Q = \frac{\Gamma}{\sqrt{\Delta^2 - \mu^2}},$$

$\Gamma = 1/\tau$ étant la largeur de raie de l'état excité (dans le cas du couplage faible $\varphi(\varepsilon_1)$ se réduit à k_1).

Substituant (24) et (25) dans (20), (23a) et (23b) on obtient

$$(26a) \quad a = \frac{1}{\mathcal{D}} \{f(\varepsilon_1)f(\varepsilon_2)(\varepsilon_1 - \Delta)(\varepsilon_2 - \Delta) + Q^2 G_{12} G_{21}\}$$

$$(26b) \quad b = \frac{1}{\mathcal{D}} \{f^2(\varepsilon_1)(\varepsilon_1 - \Delta)^2 + Q^2 G_{12}^2\}$$

avec

$$(27) \quad \mathcal{D} = |D|^2 (\varepsilon_1 - \Delta)^2 (\varepsilon_2 - \Delta)^2 = \\ = [(\varepsilon_1 - \Delta)(\varepsilon_2 - \Delta) - 5Q^2 \varphi(\varepsilon_1)\varphi(\varepsilon_2)]^2 + 9Q^2 [\varphi(\varepsilon_1)(\varepsilon_2 - \Delta) + \varphi(\varepsilon_2)(\varepsilon_1 - \Delta)]^2$$

et

$$(28) \quad G_{12} = 3f(\varepsilon_1)\varphi(\varepsilon_1) + 2f(\varepsilon_2)\varphi(\varepsilon_2).$$

Lorsque Q est petit, ce qui correspond à des vies moyennes longues, le dénominateur donne lieu à des résonances très prononcées pour $\varepsilon_1 = \Delta$ et $\varepsilon_2 \equiv \varepsilon_i - \varepsilon_1 = \Delta$. Les deux maxima correspondants dans la courbe représentant $1/\mathcal{D}$ sont nettement séparés quand $5Q^2\varphi^2 \ll ((\varepsilon_i/2) - \Delta)^2$. L'expression (26a) montre que a ne devient très grand ni pour $\varepsilon_1 = \Delta$ ni pour $\varepsilon_2 = \Delta$. Au contraire, sauf pour $\varepsilon_i = 2\Delta$, (26b) indique l'existence d'un maximum important (proportionnel à $1/Q^2$) aux environs de $\varepsilon_2 = \Delta$; pour cette valeur donc a/b est très petit. On voit alors aisément que le rapport $[\sigma'_{+0}(\varepsilon_1) + \sigma'_{0+}(\varepsilon_1)]/[\sigma'_{++}(\varepsilon_1)]$ des section efficaces différentielles est voisin de $13/2$ aussi bien pour $\varepsilon_1 = \Delta$ que pour $\varepsilon_i - \varepsilon_1 = \Delta$ (*). Pour des vies moyennes très longues les contributions principales aux intégrales intervenant dans y donné par (23) proviennent essentiellement des maxima de la courbe de résonance et d'après le résultat

(*) On vérifie d'ailleurs facilement à partir de (21) que pour $\varepsilon_1 = \Delta$ par exemple

$$\frac{\sigma'_{+0}(\Delta)}{\sigma'_{++}(\Delta)} = \frac{4}{2}; \quad \frac{\sigma'_{0+}(\Delta)}{\sigma'_{++}(\Delta)} = \frac{9}{2}.$$

précédent $y \rightarrow 0$. Dans ce cas on retrouve le rapport 13/2 pour les sections efficaces totales.

Dès que les largeurs de raie deviennent appréciables:

- 1) les contributions aux intégrales provenant des autres valeurs de ε_1 ne peuvent plus être négligées, et
- 2) les valeurs de b/a peuvent être notablement différentes de zéro, même pour $\varepsilon_1 = \Delta$, $\varepsilon_1 = \varepsilon_i - \Delta$.

Il est donc évident que la théorie élémentaire de la section 2 ne peut plus être appliquée.

Afin d'illustrer ceci nous avons effectué le calcul de y en remplaçant par des constantes les fonctions régulières $f(\varepsilon_1)$, $\varphi(\varepsilon_1)$ et q' . Ceci permet d'estimer à partir de quelles valeurs de Q la méthode élémentaire peut être appliquée (pour des énergies incidentes $\varepsilon_1 < 3\mu$). On obtient que $Q \leq 1/10$ est une condition nécessaire pour que $y \ll 1$. Cette condition peut être considérée comme suffisante, sauf pour des valeurs exceptionnelles de Δ ($\Delta \cong \mu$, $\Delta \cong \varepsilon_i - \mu$, $\Delta \cong \varepsilon_i/2$).

Le cas où Δ est de l'ordre de $\varepsilon_i/2$ présente un intérêt particulier. Le calcul précédent, appliqué à $\Delta = \varepsilon_i/2$, montre que

$$(29) \quad \frac{2}{3} \leq y \leq 1 \quad \left(\Delta = \frac{\varepsilon_i}{2}, \Gamma \text{ quelconque} \right).$$

La limite $y = 2/3$ correspond à $Q \rightarrow 0$. Ainsi, même si la vie moyenne est longue, le rapport σ_{0+}/σ_{++} est très différent de 13/2. Il est en fait voisin de 3/2. Nous reviendrons sur ce point dans la section finale.

Pour nous rapprocher plus sensiblement des conditions expérimentales ^(3,4) nous avons appliqué nos résultats à des énergies incidentes plus grandes. Evidemment le fait de ne conserver dans (11) que des états c à deux mésons devient maintenant une approximation.

D'autre part on ne peut plus considérer les fonctions $f(\varepsilon)$, $\varphi(\varepsilon)$, etc., comme des constantes et, pour les évaluer, nous avons pris les graphes de l'ordre le plus bas. Le calcul numérique montre que pour $\varepsilon_i = 1,8 \text{ GeV}$, $\Delta - \mu = 150 \text{ MeV}$, $Q = 1/3$ qui correspondent en gros aux données expérimentales, on a $y = 0,5$ et $\sigma_{+0}/\sigma_{++} \cong 2,3$. Ce rapport est sensiblement différent de la valeur 13/2.

D'autre part, comme pour le cas des énergies faibles nous avons étudié certains cas limites. En particulier lorsque $Q \rightarrow 0$, en général $y \rightarrow 0$, donc $\sigma_{+0}/\sigma_{++} \rightarrow 13/2$. Ceci reste vrai même pour des énergies incidentes très grandes ($\varepsilon_i/\Delta \rightarrow \infty$). Cependant le cas $\varepsilon_i = 2\Delta$ reste exceptionnel: lorsque $Q \rightarrow 0$, $y \rightarrow 3/2$ et non vers zéro, tout comme dans le cas des faibles énergies.

5. — Conclusions.

Dans ce qui précède nous avons étudié le rapport σ_{+0}/σ_{++} dans le cas d'un méson scalaire, le recul du nucléon étant négligé. Il est bien clair que les valeurs numériques des résultats obtenus dépendent de la théorie choisie; mais il est non moins évident que les propriétés qualitatives que nous avons déduites sont d'une généralité beaucoup plus grande, et appartiennent aussi au méson pseudoscalaire. Il nous semble donc que lorsque l'on applique des raisonnements du genre de ceux utilisés dans la méthode élémentaire de la section 2 aux cas où le niveau excité est de vie moyenne extrêmement brève, la plus grande prudence, quant aux conclusions que l'on en tire, doit être observée. D'autre part nous avons effectué un choix parmi les graphes (mais ceci pour nous rapprocher le plus possible de l'hypothèse physique du passage par un niveau excité), nous avons négligé certaines contributions des graphes (de damping) qui rendaient la solution des équations de Heitler extrêmement difficile et qui nous faisaient dépasser largement le problème que nous nous étions proposé. Les résultats déduits sont valables dans cette approximation. Il est cependant extrêmement peu probable que dans la solution exacte du problème se produisent des compensations qui redonneraient les résultats de la méthode élémentaire. Pour cette raison nous pensons que les considérations de cet article sont d'une portée générale.

D'autre part nous avons remarqué que pour certaines valeurs de l'énergie incidente, à savoir, $\varepsilon_i \approx 2\Delta$ même pour des Q très faibles, donc pour des vies moyennes longues, on ne retrouve pas les résultats de la méthode élémentaire: nous avons constaté que dans le cas $\varepsilon_i/2 = \Delta$ et $Q \rightarrow 0$, $y = 2/3$ aussi bien pour des énergies incidentes ε_i faibles que pour de grands ε_i , ceci avec des approximations dans le calcul des intégrales très différentes dans les deux cas.

Le fait que l'énergie $\varepsilon_i = 2\Delta$ soit exceptionnelle n'est pas très surprenant, si l'on se reporte à la discussion de la section 4, étant donné que les deux pôles, $\varepsilon_1 = \Delta$ et $\varepsilon_2 = \Delta$ viennent se confondre. Comme le montrerait un examen plus détaillé des formules approximatives que nous n'avons pas explicitées (leur écriture étant très lourde) les termes décrivant le damping jouent ici un rôle plus important qu'ailleurs. Il n'est donc pas dit que le choix que nous avons effectué pour les graphes décrivant le damping (cf. fig. 2) n'ai pas eu une influence sur la valeur particulière trouvée pour y . Il n'y a pas de raison cependant pour qu'un traitement plus exact de ces termes puisse redonner le $y = 0$ de la méthode élémentaire.

En fait toute la théorie de ce cas exceptionnel doit être profondément modifiée lorsque l'on tient compte du recul du nucléon qui, pour les énergies mises en jeu, n'est jamais négligeable. A l'énergie particulière $\varepsilon_i = 2\Delta$ du cas précédent correspond maintenant un intervalle d'énergies incidentes pour

lesquelles la « double résonance » est possible mais non nécessaire. Pour un tel intervalle — qui s'étend à peu près sur 150 MeV dans le cas de la production des mésons π dans les chocs mésons π -nucléon — une théorie plus développée serait indiquée.

Dans le cas général, par contre, y tend vers zéro lorsque la vie moyenne de l'état excité augmente. (22) montre que dans ces conditions le rapport σ_{+0}/σ_{++} tend vers $13/2$, valeur donnée par la méthode élémentaire. Le tableau suivant ($\varepsilon_i = 3\mu$, $\Delta - \mu = 1/9\mu$) pourra donner une idée très grossière des conditions d'applicabilité de la dite méthode aux énergies faibles.

| $Q = \Gamma/k_A$ | y | $\frac{\sigma_{+0}}{\sigma_{++}}$ | $\frac{\sigma_{+0}}{\sigma_{++}} \bigg/ \left(\frac{\sigma_{+0}}{\sigma_{++}} \right)_{\Gamma=0}$ |
|------------------|------|-----------------------------------|--|
| 1/12 | 0,85 | 0,75 | 0,115 |
| 1/36 | 0,3 | 3,7 | 0,57 |
| 0 | 0 | 13/2 | 1 |

On voit qu'il est nécessaire que Q soit très petit. Aux énergies plus grandes la situation semble plus favorable. Même à ces énergies cependant la valeur $Q = 1/3$ (qui correspond sensiblement à l'état excité $I=J=3/2$ de la diffusion méson π -nucléon) est, d'après nous, trop élevée pour que les résultats de la méthode élémentaire soient satisfaisants: ainsi pour $\varepsilon_1 = 1,8$ GeV, $\Delta - \mu = 150$ MeV, $Q = 1/3$ ⁽³⁾ on a trouvé un rapport σ_{+0}/σ_{++} deux fois plus faible que la valeur limite pour $Q = 0$ (voir section 4).

En définitive nous pensons, qu'en tout état de cause, la méthode élémentaire ne peut guère s'appliquer à des états excités dont la largeur de raie dépasse le dixième, environ, de l'énergie d'excitation.

Nous tenons à remercier le Professeur F. BLOCH pour de récentes et instructives discussions ainsi que pour son accueil au CERN à Genève. En outre nous désirons exprimer notre reconnaissance aux Professeurs L. DE BROGLIE, L. LEPRINCE-RINGUET, C. MÖLLER et A. PROCA pour leurs précieux encouragements dans nos recherches. Le présent travail a été effectué au cours de notre détachement du CNRS (Paris) au CERN.

RIASSUNTO (*)

Il lavoro si occupa dei modelli per la produzione di mesoni che fanno ricorso all'ipotesi di un nucleone intermedio eccitato X di vita breve, Es. $\pi + \mathcal{N} \rightarrow \pi + X \rightarrow \pi + \pi + \mathcal{N}$. Si discute la legittimità di trarre da tale ipotesi in unione all'indipendenza della carica conclusioni semplici e ben definite riguardanti i numeri dei mesoni creati rispettiva-

(*) Traduzione a cura della Redazione.

mente con varie cariche. Si dimostra che, contrariamente a quanto si potrebbe ritenere a prima vista, questi rapporti dipendono, in modo complesso, dall'ampiezza della riga (vita media) dello stato eccitato. Solo nel caso di un'ampiezza di riga piccola in confronto dell'energia di eccitazione detti rapporti raggiungono i valori dati dai metodi più usuali ed elementari. Si ammette la conservazione dello spin isotopico. Per semplicità i calcoli sono stati eseguiti servendosi della teoria mesonica scalare senza rineculo; le reazioni (1) e (2) si trattano a titolo di esempio. Il metodo consiste nel calcolare gli elementi della matrice T per l'intero processo, cioè senza supporre implicitamente che X sia reale. L'ipotesi di uno stato intermedio si introduce conservando solo i grafici di fig. 3. Nella soluzione delle opportune equazioni di Heitler si tiene conto degli importanti contributi di smorzamento dati dai grafici di fig. 2. Il risultato generale è dato per il nostro esempio dalla (22). La discussione di questa formula mostra che il valore limite di $\sigma\sigma_{0+}/\sigma_{++}$ può difficilmente essere una buona approssimazione a meno che la larghezza di riga Γ dello stato eccitato sia inferiore a circa 1/10 della sua energia di eccitazione Δ . Si conclude che, se X si identifica con lo stato eccitato del nucleone $I=U=3/2$, $\Delta=150$ MeV, $\Gamma=80$ MeV, il metodo elementare per il calcolo di tali rapporti non si può considerare attendibile.

Coulomb Interference in the Pion-Proton Scattering at 120 MeV (*).

L. FERRETTI, E. MANARESI, G. PUPPI and A. RANZI

Istituto di Fisica dell'Università - Bologna

G. QUARENI

Istituto di Fisica dell'Università - Padova

Istituto Nazionale di Fisica Nucleare - Sezione di Padova

(ricevuto il 9 Maggio 1955)

Summary. — A complete angular distribution for the positive pion-proton elastic scattering at 120 MeV is presented. The angular distribution deals with angles in the c.m. between 15° and 180° and is based on 545 events found in p.p. The analysis of the differential cross-section for angles $> 30^\circ$ gives as phase shifts for the partial waves, $p_{\frac{1}{2}}$, $p_{\frac{3}{2}}$

$$\alpha_3 = \mp 12^\circ, 2 \qquad \alpha_{33} = \pm 31^\circ, 4 \qquad \alpha_{31} = \mp 1^\circ, 5.$$

The choice of the sign of the phase shifts is based on the frequency of the events in the angular interval $15^\circ \div 30^\circ$. The present experiment gives evidence for the upper system of signs: the relative probability of the two systems of signs being 18.

1. — Introduction.

The interest which centres around the problem of meson-nucleon interaction is well known; our hopes of understanding nuclear forces are in fact based on our information in this connexion. At the present moment the theory of such forces is little more than at the phenomenological stage; research on the phenomena of scattering and photoproduction has in great part contributed to its construction.

(*) A preliminary account of this work was given at the 5th Rochester Conference on High Energy Nuclear Physics.

Since moreover the phenomenon of elastic scattering lends itself to an immediate intrinsic analysis, we may consider that it constitutes the most direct source of information on the above mentioned problem. In recent years, therefore, experiments and attempts at an arrangement of results obtained have been multiplied ⁽¹⁾ to the point of giving a description which is satisfying in its general outline.

In such a description it is to be admitted that the phenomena of pion-nucleon scattering can be described through the properties of two states of the pion-nucleon system, respectively of the $T=\frac{3}{2}$ and $T=\frac{1}{2}$ isotopic spin, the first of which, at moderate energies, is by far the most important. Now the scattering ($p^+ \rightarrow p^+$) is particularly interesting, because it involves the $T=\frac{3}{2}$ isotopic spin state only, while in the description of scattering ($p^- \rightarrow n^0$) and ($p^- \rightarrow p^-$) both isotopic spin states are present. In other words, the scattering amplitudes for ($p^+ \rightarrow p^+$) are those of the pure $T=\frac{3}{2}$ state, while the scattering amplitudes for the other two phenomena are linear combinations of the scattering amplitudes of the two states $T=\frac{3}{2}$ and $T=\frac{1}{2}$ indicated hereafter:

$$A_{++} = A_{\frac{3}{2}} \quad A_{-0} = \frac{\sqrt{2}}{3} (A_{\frac{3}{2}} - A_{\frac{1}{2}}) \quad A_{--} = \frac{1}{3} (A_{\frac{3}{2}} + 2A_{\frac{1}{2}}),$$

Given the said relations between the scattering amplitudes, we can understand how the properties of pion-nucleon interaction in the $T=\frac{3}{2}$ state can be established both from a study of the ($p^+ \rightarrow p^+$) scattering and from research on the independent scattering processes of the negative pions; a comparison between the results obtained in these two ways could, in the future, even furnish a quantitative indication of the degree of approximation within which the pion-nucleon system is describable through the isotopic spin. The actual experimental situation is still far from allowing a profound analysis to be made in this sense. In fact the experimental data attending the differential cross-sections originate principally from the experiments on negative pions. As a result of the difficulties which have till now been encountered in the production of intense beams of positive pions, especially at high energies, the data about the scattering of positive pions are scarcer and affected by experimental uncertainties greater than those for the negative ones.

The experiments have been carried out until now by using various techniques; counters, cloud chambers and photographic emulsions. Counters are particularly suitable for defining total cross-sections by transmission, and have shown themselves up to the present to be indispensable in experiments with

⁽¹⁾ E. H. HENLEY, M. A. RUDERMAN and J. STEINBERGER: *Ann. Rev. Nucl. Science*, **3**, 1 (1953); M. GELL-MANN and K. WATSON: *Ann. Rev. Nucl. Science*, **4**, 219 (1954); E. FERMI: *Lectures on the pions and nucleons*, in *Nuovo Cimento*, to be published.

negative mesons, especially when it is a matter of defining the differential cross-section for the process ($p^- \rightarrow n^0$). Cloud chambers and photographic emulsions can be conveniently used in a study of the processes which take place without a charge exchange, and particularly in research on the differential cross-sections of positive pions on protons.

From this last process we now know the total cross-section in the energy interval which goes from some tens of MeV up to 230 MeV ⁽¹⁾; there exist, moreover, good measurements of the differential cross-section at 40 MeV ⁽²⁾ 58 MeV ⁽³⁾ and 113 MeV ⁽⁴⁾ which have been published; results at 142 MeV ⁽⁵⁾, 150 and 170 MeV ⁽⁶⁾ and 167 and 189 MeV ⁽⁷⁾ have also been obtained. In addition there are other results at different energies based on lower statistics ⁽⁸⁾.

Our research refers to the process ($p^+ \rightarrow p^+$) at an energy of 120 MeV, the technique used being that of nuclear plates; we determined the total and the differential cross-sections, with the purpose of establishing, with a good statistical precision, the angular distribution in every angular interval, including also those near 0° and 180° , generally neglected in other experiments.

It is known that the differential cross-section for the process under examination is analysed, by admitting that the forces are short-range ones, in partial waves relative to the different angular moments, and the analysis leads to a definition of the phase shifts of the various waves.

$$S_{\frac{1}{2}}, P_{\frac{1}{2}}, P_{\frac{3}{2}}, D_{\frac{1}{2}}, D_{\frac{3}{2}}, \dots$$

At moderate energies one limits oneself normally to calculation of the first three phases, since it is supposed that only these are important, and this problem, as we know, leads to some systems of phases affected by a double indetermination. In fact, the value and sign of the phase α_3 having been fixed, there exist for the other two phases two solutions known as Fermi's solution and that of Yang, which are characterized as being $\alpha_{33} \gg \alpha_{31}$ in Fermi's and $\alpha_{31} > \alpha_{33}$ in Yang's.

Equivalent solutions are obtained then by inverting the sign of all the

⁽²⁾ I. P. PERRY and C. E. ANGELL: *Phys. Rev.*, **91**, 1289 (1953).

⁽³⁾ D. BODANSKY, A. SACHS and J. STEINBERGER: *Phys. Rev.*, **90**, 996, 997 (1953).

⁽⁴⁾ J. OREAR: *Phys. Rev.*, **96**, 1417 (1954).

⁽⁵⁾ J. J. LORD and A. B. WEAVER: Private communication.

⁽⁶⁾ CARNEGIE TECH. GROUP: Private communication.

⁽⁷⁾ H. ANDERSON: *Proc. 5-th Rochester Conference*.

⁽⁸⁾ E. C. FOWLER, W. B. FOWLER, R. P. SHUTT, A. M. THORNDIKE and W. L. WHITTEMORE: *Phys. Rev.*, **86**, 1053 (1952); A. MINGUZZI, G. PUPPI and A. RANZI: *Nuovo Cimento*, **10**, 1753 (1953); J. OREAR, J. J. LORD and A. B. WEAVER: *Phys. Rev.*, **93**, 575 (1954); G. HOME, G. GOLDBABER and L. N. LEDERMAN: *Phys. Rev.*, **93**, 554 (1954).

phases in both the previous solutions. The choice between Fermi's solution and Yang's one seems to be difficult, on the basis only of the behaviour with the energy of the phases themselves, if reference is not made here to a particular form of the theory of meson-nucleon interaction. On the other hand, from an experimental point of view, the only manner of attack seems to be that of measuring the intensity of polarisation of the scattered proton. Since such research would offer considerable difficulties, experiments in this sense have not yet been attempted, so that from this point of view the problem can be considered completely open. Some indications would point to Fermi's solution as being the more preferable.

Ambiguity in the sign, on the contrary, can be removed by observing the Coulomb interference, and that is within the limits and one of the aims of the present experiment. A definition of the signs could also be reached, independently, through experiments on the polarisation of the scattered proton; the sign of polarisation in fact is connected with the sign of the phases.

2. - Exposure.

A stack of twenty «stripped» emulsions, of the thickness of $600\ \mu$ and measuring $7.11\ \text{cm}^2$, was exposed to a beam of 135 MeV positive pions (nominal energy) produced by the Chicago synchrocyclotron (*). The particles entered one of the faces of the emulsion stack perpendicularly, interesting a central zone of about $41.2\ \text{cm}^2$. The flux of tracks appears, on an average, to be 50 000 per cm^2 ; we thought of limiting the flux to this figure in order to ensure that we should be able to observe the events satisfactorily: the poor density of the tracks is largely compensated for by the ample volume of gelatine exposed. Only some of the central emulsions of the stack were partially explored, from 0.5 cm from the entry edge, for a length of 3 cm in the beam direction.

3. - Analysis of the Events.

The problem of the identification of the events $\pi + \text{H}$ in photographic emulsion has already been illustrated and discussed in previous work (⁹). It takes place through an analysis of kinematics, which, dealing with an elastic collision between two bodies, is particularly simple. There exist, that is to say,

(*) Thanks to the interest taken by Professor M. SCHEIN and to the kind collaboration on the part of Doctors D. HASHKIN and J. OREAR.

(⁹) G. GOLDBABER: *Phys. Rev.*, **89**, 1187 (1953).

well-defined relations between the ϑ and φ scattering angles, defined respectively by the directions: incident pion-scattered pion, incident pion-diffused proton. The three tracks must be coplanar. Other relations connect the angles with the post-collision pion kinetic energy or the diffused-proton energy.

The spurious events have been separated from the true ones in a statistical manner by considering the deviation from coplanarity and the relation between the scattering angles. We determined the two definite planes: one from the incident-pion and scattered-pion tracks and the other from the incident-pion and proton ones. The $\Delta\gamma$ angle between the said planes was assumed as the deviation from coplanarity. For the deviation from the angular relation we took the distance d of the experimental point from the curve which represents the above-mentioned relation. The deviation distributions were approximated with some gaussian ones, of which we determined the mean quadratic deviations. The deviations from coplanarity assembled reunited in three different groups according to the ϑ angle, namely for $0 < \vartheta \leq 60$, $60 < \vartheta < 120$, $120 < \vartheta \leq 180$. The mean quadratic deviations of the first and third group appeared greater than those of the second group, as was to be expected, since coplanarity is more difficult to determine when the tracks form among themselves angles near 0° or 180° . Each deviation was divided by the corresponding average quadratic deviation, finding so the distribution of the measurements $m = \sqrt{(\Delta\gamma/\overline{\Delta\gamma})^2 + (d/\overline{d})^2}$. Obviously the mean quadratic deviation of this distribution must be $\sqrt{2}$. All the events for which m appeared less than $\sqrt{2}$ were assumed as true collisions on the hydrogen; those with $2\sqrt{2} < m < 4\sqrt{2}$ dubious, and those with $m > 4\sqrt{2}$ spurious.

The use of «stripped» emulsions made it possible to determine the proton ranges: by passing from one emulsion to another the tracks can in fact be followed until they stop. From a range so measured and from the scattering angles, it is possible to deduce the pion kinetic energy before collision. If this value appears compatible with the energy of the pions belonging to the beam, we may consider that the event is a true collision on the hydrogen, otherwise it must be considered spurious. This analysis, therefore, furnishes a new separation criterion which is different from the previous one, which, however, showed itself to be satisfactory except for a few exceptions; it allowed us, moreover, to eliminate some doubtful cases.

We would point out that the precision improves as the ϑ angle increases and the φ one decreases, in the cases, that is to say, in which the energy of the proton and, in consequence, its range, are at their maximum. The use of «stripped» emulsions then appears very convenient, if one wishes to determine the energy distribution of the pions at their interaction with the protons; such a distribution is shown in Fig. 1. By adding in each case the energy loss through ionization along the path covered in the emulsion prior to the collision, we obtain the entry energy spectrum which is shown in Fig. 2.

We found the distributions by assigning to the individual events rectangles with equivalent bases equal to the measurement indetermination; these are

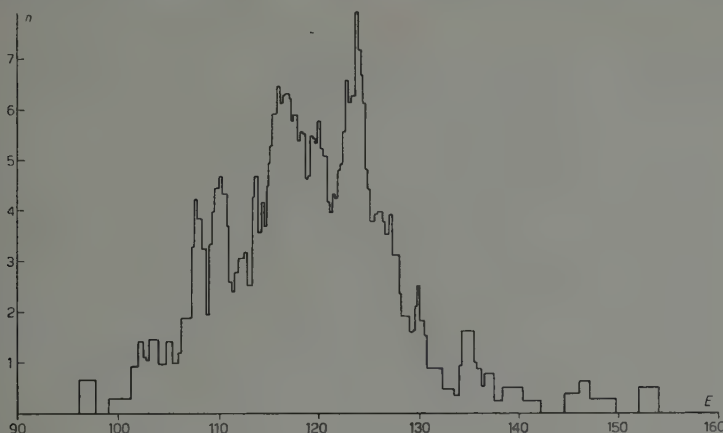


Fig. 1. — Energy distribution of pions at the interaction.

based on 122 events chosen from those having their ϑ angle greater than 60° , i.e. those which lend themselves to the most exact determination of the incident pion energy. The mean value of the interaction energy appeared as

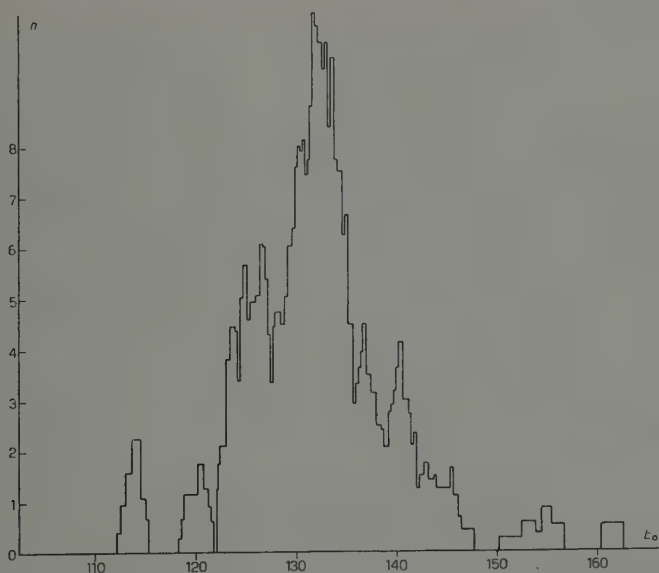


Fig. 2. — Energy distribution of pions at the entry in the plates.

120 MeV, with that of the beam entry energy as 132 MeV which is in agreement with the nominal value of 135 ± 5 MeV.

4. - Scanning Method.

The scanning was carried out by five observers with the « area scanning » method through lines parallel to the beam direction for a length of 3 cm and separated from one another by 200 μ intervals. Immersion objectives were used; the total magnification was 440, and the field diameter 350 μ . The total surface explored was 18.5 cm², of which about 70% was seen twice by different observers. All the interactions found were observed paying particular attention to the black or grey tracks originating in the gelatine, in order to ascertain if they were associated or not with the scattering of a meson. Nevertheless, it is possible that some events may have been missed. We therefore tried to estimate these losses in the most exact manner possible, and to single out the characteristics of those events, for which the inefficiency of observation is very great. The following characteristics resulted:

- a) very short-recoiled proton track; that is to say, a very small ϑ angle;
- b) the scattering angle ϑ near 180° and so the φ one very near 0°; in certain cases the two tracks left by the pion, especially if overlapping or overlapping casually on any one track of the beam, can simulate a grey track which continues the proton's;
- c) the event's position in the gelatine very near one of the surfaces;
- d) collision plane near the perpendicular one at the emulsion surfaces.

The characteristics c) and d) can be associated with the previous ones.

We therefore examined the frequency of the events in function of the coordinate z perpendicular to the emulsion plane, and constructed various histograms representing the collision-plane frequencies for different scattering groups characterized by different measurements of ϑ . For the events with a very small ϑ ($\vartheta < 30^\circ$), a valuation of observation inefficiency turned out rather uncertain owing to their being few in number. To obviate this we took into consideration all the events which, in our opinion, present the same difficulties of being seen; events characterized, that is to say, by a short black track joined to one or two thin tracks.

The analysis, which has been developed up to the present, permits us to affirm that, if scatterings with $\vartheta < 12^\circ$ are not taken into consideration, there do not exist observation efficiency differences for the different scattering groups; the correction factor, which is essentially due to the minor efficiency noticed in the finding of those scatterings having the characteristics mentioned under point d), emerges as 1.16 ± 0.03 .

Another type of control was carried out in order to ascertain the finding

of scatterings with ϑ very near 180° ; an observation was made along lines perpendicular to the beam direction having a distance of 5 mm between them in the «area» zones previously observed. All the grey or black tracks sighted in the microscopic field, which formed an angle of less than 25° with the beam direction, were followed in the direction opposite to the beam one until they left the plate or appeared as emerging by an interaction. In this manner we found many $\pi+H$ events, which were naturally characterized by a large ϑ angle. All, except seven, had already been found in the «area scanning». Taking into account that the tracks of the incoming pion and scattered pion were practically overlapping, only three of the seven, we believe, could easily be found in the «area scanning».

As can be seen from the double observation, if the events having the characteristics cited in points *a*), *b*), *c*) and *d*) are not taken into account, it can be deduced that the efficiency of each observer was on an average 95%, and that of the double observation practically total. The correction factor, in which all the loss causes are globally taken into account, and according to which the frequencies observed must be multiplied, appears as 1.17 ± 0.05 .

5. — Results.

a) Total Cross Section. — When the number of given atoms per cm^3 is known, the determination of the total cross-section leads back to that of the mean free path for the collision on the hydrogen in the emulsion. Various methods can be followed. The most direct one is that of scanning along the track. The mean free path, however, being very long, about 3 metres, it is very laborious to reach good statistical accuracy in this manner. We followed the beam tracks for a total length of 32 metres finding 9 $\pi+H$ events. Taking into account the beam impurity due to a 5% of μ -mesons and a 1% of electrons, the value of λ found is:

$$\lambda_1 = 334 \pm 111 \text{ cm.}$$

A second method is based on the number of events found per area, corrected for observation inefficiency, and on the measurement of the total track length in the zone explored, namely 118.540 cm. In this manner we got:

$$\lambda_2 = 284.6 \pm 18 \text{ cm.}$$

A third way is that of measuring the mean free path for interaction with visible excitations by means of scanning along the track, and of comparing the $\pi+H$ number with the stars, due to beam particles and observed per area

in the same zone. The value obtained is:

$$\lambda_3 = 287.4 \pm 32.5 \text{ cm},$$

The hydrogen content in the gelatine was furnished by the producing firm; it measured $n_H = 3.37 \cdot 10^{22}$ atoms per cm^3 . This value is, however, uncertain, inasmuch as the hydrogen content varies with emulsion humidity, which in turn depends on that of its surroundings. Unfortunately we do not know the humidity conditions of the emulsions when exposed. Using the datum furnished by the producers, we found the following cross-section values corresponding to the three methods:

$$\sigma_1 = 88.6 \pm 29.5 \quad \sigma_2 = 104.3 \pm 6.6 \quad \sigma_3 = 103.2 \pm 11.7.$$

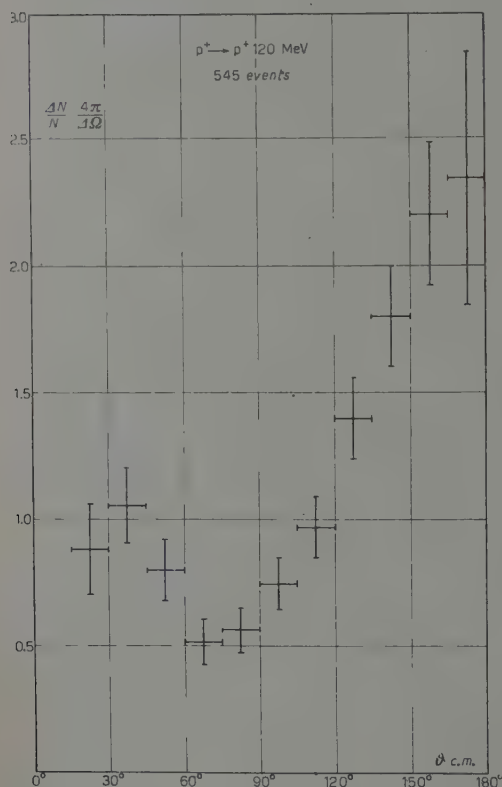


Fig. 3. - Angular distribution for the $p^+ \rightarrow p^+$ process at 120 MeV.

The weighted average of these values is:

$$\sigma = 103.5 \pm 5.6.$$

If compared with the results of the experiments based on other methods which give 95 mb for σ at this energy, this figure seems to us to be a little too high. It is probable that the discrepancy depends on an undervaluation of our's of the number of centres in the sense mentioned above. Therefore we use in what follows 95 mb for the total cross-section.

b) Angular Distribution and Differential Cross-Section. - The angular distribution is shown in Fig. 3. It is based on 545 events, grouped in 15° intervals of the ϑ angle which is calculated in the centre of the mass. The first interval, 0° - 15° , was disregarded; the finding and identification of such

events must be considered as very uncertain, owing to recoil-proton track shortness.

We brought the relative frequencies $\Delta\nu/\nu$, divided by the solid relative angle $\Delta\Omega/4\pi$ to ordinates. The errors are due to statistical uncertainties.

TABLE I.

| $\vartheta_{c.m.}$ | ν | $d\sigma/d\Omega$ |
|--------------------|-------|---------------------------|
| 15° - 30° | 24 | 6.65 ± 1.36 mb/sterad |
| 30° - 45° | 46 | 7.99 ± 1.17 |
| 45° - 60° | 45 | 6.06 ± 0.90 |
| 60° - 75° | 34 | 3.88 ± 0.68 |
| 75° - 90° | 40 | 4.26 ± 0.68 |
| 90° - 105° | 53 | 5.67 ± 0.76 |
| 105° - 120° | 64 | 7.33 ± 0.91 |
| 120° - 135° | 79 | 10.58 ± 1.21 |
| 135° - 150° | 78 | 13.61 ± 1.52 |
| 150° - 165° | 60 | 16.65 ± 2.12 |
| 165° - 180° | 22 | 17.73 ± 3.78 |

The differential cross-section appears in Fig. 4. Here the errors take into account the statistical uncertainties and that of the correction factor. Table I reports the frequencies observed, those corrected and the corresponding cross-sections for every angular interval.

6. - Discussion of Results and Conclusions.

The discussion of the results obtained is summarized in an analysis of the angular distribution in terms of the phase shifts of the different partial waves. For this purpose it is necessary to prefix the partial waves which we want to use, and this takes place according to different criteria. If the meson-nucleon forces extend themselves within a distance of the order of the Compton wave length of the (\hbar/mc) meson, a general argument of quantum mechanics would suggest that the analysis in the *S* and *P* waves alone

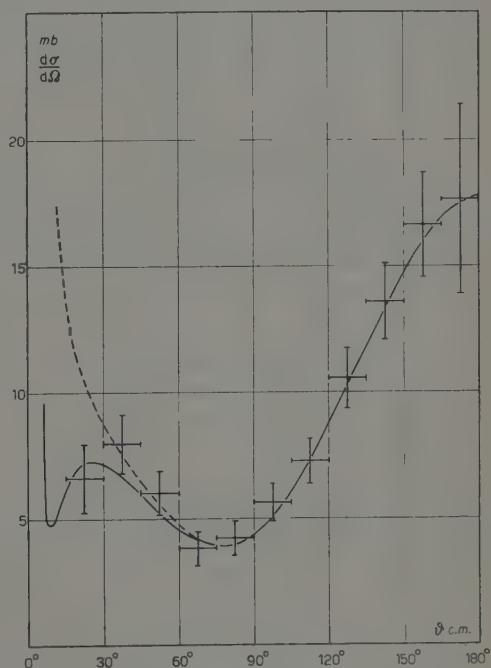


Fig. 4. -- Differential cross section for the $p^+ \rightarrow p^+$ process at 120 MeV; the two curves represent two solutions with opposite sign in the phase shifts.

ought still to be sufficient at our energy, and that then a contribution of higher momenta should be smaller. On the other hand, the peculiar behaviour of the S and P waves, which appeared after the analysis had been carried out in these terms, renders the validity of this argument very doubtful. In effect the behaviour of the differential cross-section should indicate directly if the D wave contributes appreciably beyond the S and P waves, but the experimental errors are still great, for the representations with or without the D wave to have comparable reliability.

We shall begin then with an analysis of the S and P waves alone, discussing thereafter the consequences of the D terms. To do this let us proceed as follows.

An immediate verification shows that the differential cross-section for angles greater than 45° is not greatly influenced by Coulomb terms, and, therefore, we can represent it with a quadratic function of $\cos \vartheta$ of the type:

$$f(\vartheta) = a_0 + a_1 \cos \vartheta + a_2 \cos^2 \vartheta ;$$

a_0 , a_1 , a_2 , being the coefficients, calculable starting from the experimental values with the usual methods of the least squares, and using 96 mb for the total cross-section

$$a_0 = 4.38 \pm 0.21, \quad a_1 = 3.93 \pm 0.36, \quad a_2 = 9.53 \pm 0.56.$$

Let us now turn to the relations, which connect the coefficients with the scattering amplitudes and phase shifts

$$\begin{aligned} a_0 &= |s|^2 + |p_+ - p_-|^2 \\ a_1 &= 2I(s|2p_+ + p_-) \\ a_2 &= |2p_+ + p_-|^2 - |p_+ - p_-|^2 \end{aligned} \quad \left\{ \begin{array}{l} s = \exp[i\alpha_s] \sin \alpha_s \\ p_+ = \exp[i\alpha_{3s}] \sin \alpha_{3s} \\ p_- = \exp[i\alpha_{31}] \sin \alpha_{31} \end{array} \right.$$

where $I(a/b) = (ab^* + a^*b)/2$ represents interference terms, and let us calculate the Fermi type phase shifts by Askin and Vosko's graphic method ⁽¹⁰⁾. So we get

$$\alpha_s = \mp 12^\circ.2, \quad \alpha_{3s} = \pm 31^\circ.4, \quad \alpha_{31} = \mp 1^\circ.5.$$

With these phase shifts, let us construct the complete solutions with the Coulomb terms, corresponding to the two alternatives in the sign, by using Solmitz's formulas ⁽¹¹⁾, and let us compare them with the differential cross-sections in

⁽¹⁰⁾ J. ASKIN and S. H. VOSKO: *Phys. Rev.*, **91**, 1248 (1953).

⁽¹¹⁾ F. T. SOLMITZ: *Phys. Rev.*, **94**, 1799 (1954).

the whole angular interval. This is done in the figure, which shows very clearly that the depression of the cross-section in the forward direction favours a particular choice of the signs, namely

$$\alpha_3 = -12^\circ.2, \quad \alpha_{33} = +31^\circ.4, \quad \alpha_{31} = -1^\circ.5.$$

To put this choice on a quantitative basis, we can calculate, on the angular interval 15° - 30° , which was previously rejected in looking for the phases, the frequency which may be expected from the two solutions, which are respectively 26 and 38. Since the frequency which was effectively observed is $n=24$ we can deduce therefrom that the probability ratio of the two solutions is $P_{26-24}/P_{38-24} \cong 18$ in favour of the solution which we had indicated.

At this stage in our problem we do not consider it opportune to proceed to a more refined mathematical analysis of the angular distribution, which only much more exact experimental data could render necessary.

Let us now pass to an analysis of the angular distribution, including the D waves. If in such a case the first interval is always excluded, the angular distribution may be represented with a function of the type

$$f(\vartheta) = a_0 + a_1 \cos \vartheta + a_2 \cos^2 \vartheta + a_3 \cos^3 \vartheta + a_4 \cos^4 \vartheta,$$

where the individual coefficients are united to the individual scattering amplitudes and phase shifts by the relations:

$$a_0 = |s|^2 + |p_+ - p_-|^2 + \left| \frac{3}{2}d_+ + d_- \right|^2 - 2I(s \left| \frac{3}{2}d_+ + d_- \right|)$$

$$a_1 = 2I(s |2p_+ + p_-| - 2I(\left| \frac{3}{2}d_+ + d_- \right| |2p_+ + p_-|) + 6I(p_+ - p_- |d_+ - d_-|)$$

$$a_2 = |2p_+ + p_-|^2 - |p_+ - p_-|^2 + |9|d_+ - d_-|^2 - 6 \left| \frac{3}{2}d_+ + d_- \right|^2 + 6I(s \left| \frac{3}{2}d_+ + d_- \right|)$$

$$a_3 = 6I(2p_+ + p_- \left| \frac{3}{2}d_+ + d_- \right|) - 6I(p_+ - p_- |d_+ - d_-|)$$

$$a_4 = 9 \left| \frac{3}{2}d_+ + d_- \right|^2 - 9 |d_+ - d_-|^2,$$

where, in addition to the S and P waves, we have for D waves

$$d_+ = \exp[i\delta_{35}] \sin \delta_{35}, \quad d_- = \exp[i\delta_{33}] \sin \delta_{33}.$$

The terms containing differences between the scattering amplitudes are spin-flit terms.

We have shown that the data are globally well representable with the S and P waves alone; it will then not be easy to discover the eventual contribution of the D wave. In these conditions we consider that the probability of having an indication must essentially be based on an analysis of the data

concerning the angular interval near 180° ; in our case we do not consider the depression which we observed in the angular distribution near 180° significant, since the experimental errors are too great; we consider, however, that, from an experimental point of view, progress might be made in the definition of the behaviour in this region.

If the contribution of the D wave were considerable, the situation in the angular interval near 0° should, of course, be rediscussed, because there the contribution of the D waves could alter the conclusions on the effect of the Coulomb forces. In conclusion of this first analysis we can in consequence affirm that, if the angular distribution be representable in terms of the partial S and P waves, then the present experiment gives evidence in favour of the choice of the phase signs. In the Orear experiment at 113 MeV the situation was similar to our's, and the conclusions the same, for what concerns the Coulomb interference.

Our research is continuing in the hope of bringing other contributions to the problem just discussed.

Our thanks are due to Dr. G. FERRARI, R. GESSAROLI, S. STANTIC, E. PEDRETTI, A. STANGHELLINI who friendly collaborated with us in various stages of this research.

Thanks are due also to M. BARBIERO, F. CIANCABILLA, S. MARCHETTI, C. MARCHI, A. MONREALE, C. SOAVE for excellent scanning of the plates.

RIASSUNTO

In questo lavoro viene presentata una distribuzione angolare completa per il fenomeno di diffusione elastica pione positivo-protone a 120 MeV. La distribuzione interessa angoli nel c.m. che vanno tra 15° e 180° ed è basata su 545 eventi trovati in lastre fotografiche. L'analisi della sezione d'urto differenziale per angoli $> 30^\circ$ in termini di sfasamento delle onde parziali, s , $p_{\frac{1}{2}}$, $p_{\frac{3}{2}}$, porta ai valori seguenti

$$\alpha_3 = \mp 12^\circ.2 \quad \alpha_{33} = \pm 31^\circ.4 \quad \alpha_{31} = \mp 1^\circ.5.$$

La scelta del segno delle fasi è fondata sulla frequenza degli eventi nell'intervallo angolare $15^\circ \div 30^\circ$. Il presente esperimento dà evidenza per il sistema superiore di segni: la probabilità relativa dei due sistemi di segni è 18.

NOTE TECNICHE

A Method for Stripping Nuclear Emulsions from their Glass Supports and Remounting Them.

C. DAHANAYAKE

H. H. Wills Physical Laboratory, University of Bristol

(ricevuto il 4 Aprile 1955)

Summary. — A method for stripping nuclear emulsions from their glass supports and remounting them is described. The stripping is effected by immersing the plate in a weak solution of hydrofluoric acid in sodium acetate buffer. After careful washing the remounting is carried out as in the case of new stripped emulsions.

1. — Introduction.

A large part of emulsion work carried out these days involves the use of stacks of stripped emulsions. Consequently, it is normal practice to handle large numbers of emulsions mounted on glass supports, in the various operations of processing, alignments of the stack, and in the examination and measurement of events on the microscope. There is also a trend towards using larger and larger stacks, both in the number of emulsions and the linear dimensions. In view of this, it is not unexpected that, occasionally, in spite of every care taken, some plates are broken. A fractured glass support is a source of constant trouble, since the optical qualities are impaired; there is danger of the emulsion tearing itself away along the cracks and the correct registering of the plate is often difficult and sometimes impossible. Quite obviously such a plate represents a barrier which greatly reduces the value of the stack. If a plate is fractured during the processing, the cutting of the plate is liable to introduce further fractures. In view of these it was thought desirable to develop a method for stripping the emulsion sheets from their glass supports and remounting them on new ones.

Such a method is also of value in cases when, in an otherwise satisfactory plate, there are unduly heavy deposits of silver on the emulsion surface nearest the glass support as a result of fogging due to light or pressure. These silver deposits can only be removed by rubbing the appropriate surface with alcohol or similar suitable agent. To be of any value, the method of stripping should have little or no effect on the developed characteristics of the emulsion being treated, i.e. its linear dimensions, the size and number of grains on tracks passing through it and on the level of distortion.

We have, in this laboratory, tried two methods which would appear to fulfil these conditions to a fair degree. As will be seen, the second method is by far the simpler and more satisfactory.

2. – Water Vapour Method.

In this, the plates were rested on a platform, the level of which was above that of the water contained in a large trough; the whole was then covered with a large glass bell jar. The temperature of the water was maintained constant at various different values and the plates were examined at regular intervals.

The results are shown in the table below:

| Temperature (deg. C) | Observations |
|-------------------------|--|
| 25° | Slight swelling; no stripping even after 6 hours. |
| 30° | Same as 25°. |
| 35° | After 6 hours the emulsion, when eased out at the edge, came off the glass. There is definite stretching of the emulsion but it appeared to reach normal size when released. |
| 40° | Emulsion dissolves. |

Even at 35°, when the emulsion could be stripped off, there was a slight tendency for the emulsion to soften round the edges. Moreover, a certain temporary distension of the strip was unavoidable since considerable tension had to be applied to force it off the glass. This method was therefore not pursued further.

3. – The Hydrofluoric Acid Method.

This was based on the idea that if one could dissolve, even to a small extent, the glass at the glass-emulsion interface without damaging the emulsion, then the emulsion ought to come free. The gelatine itself is not affected by hydrofluoric acid in any serious way. The hydrofluoric acid would be expected to attack the surface layers of the silver grains to form silver fluoride which is soluble in warm water, but in a weak acid solution this effect was presumed negligible. These conclusions were borne out by the experiments. As pointed out to us by Mr. WALLER of Ilford Ltd., this method is a known process in commercial photographic practice.

The initial experiments were performed with an aqueous solution of the acid. The plates were immersed in .1% and .2% solutions (by volume) of the acid in water. After about 6 hours of soaking, the emulsion came off the

glass with practically no effort. Though otherwise satisfactory, the excessive swelling of the emulsion in the acid solution left it permanently distended. Less concentrated solutions simply prolonged the time of soaking without preventing distension. It might be possible to restrain swelling by the use of a high concentration of sodium sulphate, but there might then be some difficulty in removing the sulphate. However, the swelling of the gelatine can be minimized by working throughout at constant pH of ~ 4.8 . Hence, in the later experiments a sodium acetate buffer solution with a pH of 4.63 was used instead of water for the preparation of the acid solution.

The buffer was prepared as follows:

| | |
|--|--|
| Acetic acid | 12.0 g ($\sim 12 \text{ cm}^3$ conc.) |
| Sodium acetate | 16.4 g |
| Distilled water to make up to a litre. | |

The experiments carried out using the buffer were successful and the following is an account of the method finally adopted.

The plate was soaked in .2% solution (by volume) of hydrofluoric acid in sodium acetate buffer. The temperature was that of the room which did not vary much from 12°C . After about eight hours, the emulsion could be peeled off the glass very easily. If the slightest tendency to adhere persisted the soaking was continued. Once the wet emulsion is free of the restraining effect of the glass support it starts to distort itself giving way to the strains always present in it. Because of this, washing for a prolonged period before remounting was not practicable. This was so even when washing in the pure buffer solution was tried. The stripped emulsion was rinsed in the pure buffer and remounted in the usual manner adopted in this laboratory — (the mounting being carried out in a warm gelatine solution at $\sim 25^\circ\text{C}$ on a sheet of glass which had been previously cleaned with a weak solution of teepol and rinsed in distilled water). The wet emulsion tended to slip on the glass and therefore had to be lightly held down in the middle and squeezed towards the edges. The plate was quickly mopped with filter paper and left to dry in air for three hours.

The fairly dry plate was then washed in the pure buffer solution and transferred to fresh buffer every six hours. After 24 hours of this treatment most of the acid was removed. The washing was then continued in water at low temperature and with a slow rate of flow. After 24 hours in water, the plate was transferred to a 4% solution of glycerine and left for six hours. It was finally mopped with filter paper and dried as usual.

4. — Chemical Effects.

The effect of the hydrofluoric acid and the buffer solution on the silver grains was examined. For this purpose grain counts of tracks before and after stripping were taken. There was no indication that harmful chemical changes had taken place. In another case, a plate treated with the acid-buffer solution was fixed in hypo before washing. Again no change was observed. In one test plate no glycerine was used in the final washing. It might be expected

that, if the adhesive properties of the remounted strip were not satisfactory it would peel off from the glass support in drying. Instead, when completely dried out, as in the case of ordinary glass backed plates, the emulsion on stripping brought off slivers of glass with it indicating strong adhesion to the support.

No serious tests on scattering in the treated plates were made. It is doubtful if one can attach much significance to scattering measurements made on a strip which has undergone such diverse treatment. This, however, remains to be investigated fully and like distortion in normal processing is a feature characteristic of each emulsion sheet and stack, and the handling.

The plates stripped and remounted as described had negligible change in dimensions. The registering with other plates of the same stack is accurate, and it is possible to follow tracks through it as in a normal plate.

It might also be feasible to strip emulsion sheets from their supports and mount them upside down on new glass plates. In this position it is possible to wipe off silver deposits which were present on the surface originally in contact with the glass. After wiping off the black silver marks, the stripping and remounting upside down procedure can be repeated so that the emulsion sheet is back again in its normal position.

It is essential to point out that the emulsion is necessarily in a weak condition when wet, particularly when being stripped and when off its support, and must be treated with especial care.

It must also be emphasized that hydrofluoric acid is extremely poisonous and should be handled with great care; in particular the hands should not be wetted by it.

The author wishes to thank Mr. C. WALLER of Ilford Ltd., for helpful suggestions and criticism and Professor C. F. POWELL for kindly extending the hospitality of his laboratory. He is deeply indebted to Dr. M. G. K. MENON for suggesting the experiment and for the constant encouragement and criticism received.

RIASSUNTO (*)

Si descrive un metodo per staccare le emulsioni nucleari dai loro supporti di vetro e poi rimontarle. Il distacco si ottiene immergendo la lastra in una soluzione diluita di acido fluoridrico tamponata con acetato di sodio. Dopo lavaggio accurato il nuovo montaggio si esegue come per le emulsioni senza supporto nuove.

(*) Traduzione a cura della Redazione.

Una miscela di gas per il riempimento di camere di ionizzazione rapide, poco sensibile a contaminazioni di ossigeno.

U. FACCHINI e A. MALVICINI

Laboratori CISE - Milano

(ricevuto il 16 Aprile 1955)

Riassunto — Viene studiato il comportamento di argon e azoto molto puri e di miscele argon-azoto come gas di riempimento di una camera di ionizzazione a raccolta di elettroni. In particolare viene rilevato che mentre il comportamento dell'argon e dell'azoto è molto critico a piccole tracce di ossigeno, miscele contenenti 96-98% di argon e 4-2% di azoto hanno un funzionamento pressochè indipendente dalla presenza di tracce di ossigeno fino a percentuali di circa 0,5%. Viene studiato il modo di impiego di queste miscele e mostrato come con esse si possano adoperare rapidamente e semplicemente le camere di ionizzazione rapide.

1. — È noto che le camere di ionizzazione a raccolta di elettroni sono uno strumento di buona precisione per la rivelazione e per la misura dell'energia di particelle α , di protoni e in generale di particelle pesanti ⁽¹⁾.

Il maggior inconveniente, che si ha nel loro uso, sta nel fatto che, per ottenere una completa raccolta di elettroni, il gas di riempimento della camera deve essere molto puro e cioè non contenere, neppure in tracce, gas o vapori che abbiano una grande probabilità di catturare gli elettroni.

Fra questi gas il più comune è l'ossigeno, la cui probabilità di cattura è

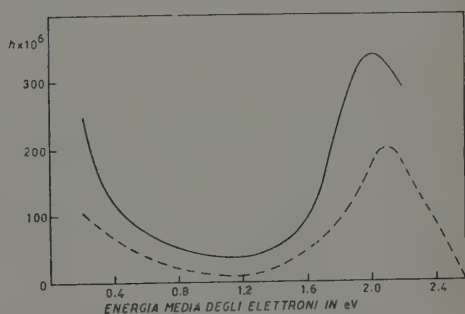


Fig. 1. — Probabilità di cattura di elettroni in ossigeno (h) riferita ad una singola collisione. ----- Risultati di BRADBURY; — Risultati di HEALEY e KIRKPATRICK.

⁽¹⁾ B. ROSSI e H. STAUB: *Ionization chambers and counters* (New York, 1949).

stata misurata per energie degli elettroni fino a qualche eV da BRADBURY ⁽²⁾ e da BROSE ⁽³⁾; per comodità riportiamo in fig. 1 i risultati ottenuti in queste esperienze. Al disotto di 0,4 eV la probabilità di cattura ha valori piuttosto elevati ed è decrescente al crescere dell'energia degli elettroni fino a raggiungere un largo minimo intorno a 1 eV; a 2 eV c'è quindi un massimo pronunciato.

Scopo di questo lavoro è quello di studiare la criticità a tracce di ossigeno

di una camera di ionizzazione del tipo indicato sopra, nel caso che i gas di riempimento siano rispettivamente l'argon e l'azoto, e particolari miscele di questi due gas.

La camera di ionizzazione usata, del tipo a elettrodi piani con griglia, è simile ad una descritta in un precedente lavoro ⁽⁴⁾. La distanza fra l'elettrodo negativo e la griglia è di 23 mm e quella fra la griglia e l'elettrodo collettore è di 7 mm. La camera viene inserita in un circuito di purificazione e di miscelazione dei gas in studio (fig. 2). Dopo una prolungata vuotatura a 10^{-5} mm Hg, essa viene riempita con la miscela desiderata.

Sull'elettrodo negativo è collocata una sorgente

di uranio naturale depositata in uno strato sottile rispetto al percorso delle particelle α . A questo elettrodo è applicata una tensione negativa variabile da 200 a 4000 V, tensione che nel seguito è riferita come tensione totale. La griglia della camera è mantenuta ad una tensione negativa pari a circa $2/3$ della tensione totale.

L'elettrodo collettore è collegato ad una usuale catena di amplificazione avente un tempo di salita di $0,6 \mu s$ e la misura consiste nel rilevare gli impulsi delle particelle α dell'uranio, di misurarne l'ampiezza ed in taluni casi di analizzare il loro spettro di ampiezza con un analizzatore a 99 canali.

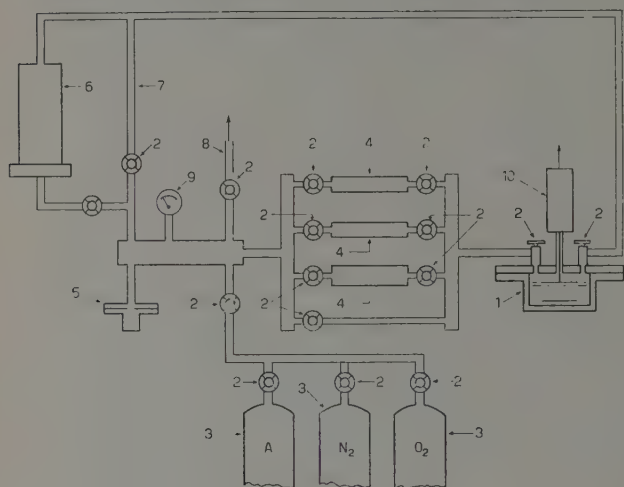


Fig. 2. - Apparecchiatura sperimentale. 1) Camera di ionizzazione. 2) Rubinetti metallici. 3) Bombole di argon, azoto e ossigeno. 4) Serbatoi per la miscelazione dei gas. 5) Vacuometro di Penning. 6) Fornace di purificazione. 7) Colonna riscaldata per provocare la rapida miscelazione dei gas. 8) Attacco della pompa a vuoto. 9) Manometro. 10) Preamplificatore.

⁽²⁾ N. E. BRADBURY: *Phys. Rev.*, **44**, 883 (1933); *Journ. Chem. Phys.*, **2**, 827 (1934).

⁽³⁾ H. L. BROSE: *Phil. Mag.*, **50**, 536 (1925); R. H. HEALEY and KIRKPATRICK, riportati in R. H. HEALEY and J. W. REED: *The Behaviour of Slow Electrons in Gases* (Sidney, 1941).

⁽⁴⁾ U. FACCHINI e E. GATTI: *Nuovo Cimento*, **7**, 589 (1950).

2. - Le fig. 3 e 4 mostrano le curve che danno l'ampiezza degli impulsi di particelle α in funzione della tensione totale, per argon e per azoto. L'argon e l'azoto, contenuti in due bombole, hanno purezza dell'ordine del 99,99%; l'argon è ulteriormente purificato nella fornace contenente una lega di calcio magnesio inserita nel circuito di fig. 2 (5). Dopo questa purificazione si ha un aumento dell'ampiezza degli impulsi a tensioni molto basse, minori di 300 V. La pressione usata in tutte le misure è di 3 atmosfere, in modo che le particelle α esauriscano il loro percorso nello spazio fra l'elettrodo a tensione e la griglia.

Si vede dalle curve delle fig. 3 e 4 che, come è noto, argon e azoto molto puri danno una buona collezione di elettroni in un vasto campo di tensioni.

Nelle stesse figure sono indicate le ampiezze di impulsi a varie tensioni ottenute con argon e azoto, ai quali è aggiunto ossigeno nella proporzione dello 0,05 % (curve b).

Nel caso dell'azoto, l'effetto dell'aggiunta di ossigeno è quello di impedire la raccolta degli elettroni a bassi valori della tensione applicata. Ciò corrisponde al fatto che a questi valori della tensione l'energia degli elettroni è inferiore a 0,5 eV, cioè è nella zona in cui l'ossigeno presenta una grande sezione di cattura. Aumentando la tensione l'energia degli elettroni aumenta (1,6) e la cattura diminuisce.

Nell'argon invece si vede che l'ossigeno cattura per lo più gli elettroni a tensioni superiori a 1000 V e che aumentando la tensione la cattura aumenta. Questo comportamento si spiega bene considerando che nell'argon l'energia

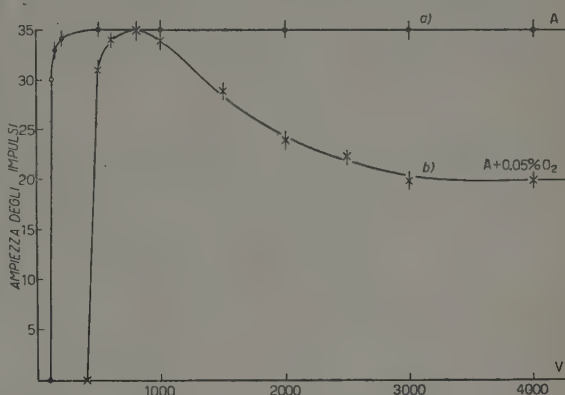


Fig. 3. - Ampiezza degli impulsi di particelle α alle diverse tensioni totali applicate alla camera. La tensione di griglia è negativa e circa i due terzi di quella totale. Gas di riempimento: curva a) argon 99,99%; curva b) lo stesso argon + 0,05 % di ossigeno. Pressione: 200 cm Hg.

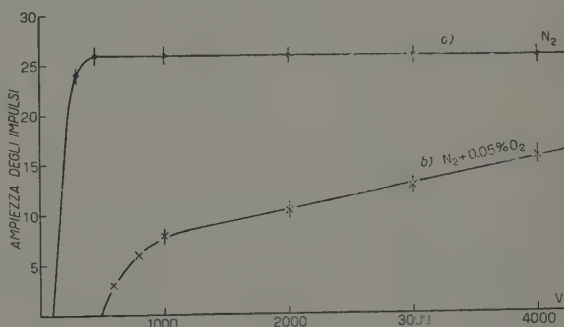


Fig. 4. - Ampiezza di impulsi di particelle α in funzione della tensione applicata. Gas di riempimento: curva a) azoto 99,99%; curva b) lo stesso azoto + 0,05 % di ossigeno. Pressione: 200 cm Hg.

(5) L. COLLI e U. FACCHINI: *Rev. Sci. Inst.*, 23, 39 (1952).

(6) R. H. HEALEY e J. W. REED: *L. c.*

degli elettroni è, nelle nostre condizioni di campo elettrico e di pressione, già molto elevata, 1 eV e più, ad una tensione di 1000 V e aumenta rapidamente al crescere della tensione (^{1,6}). Aumentando la tensione gli elettroni raggiungono energie intorno a 2 eV dove l'ossigeno ha un massimo di probabilità di cattura e a seguito di ciò si ha una riduzione dell'ampiezza degli impulsi.

Questi risultati dimostrano che sia l'azoto che l'argon sono piuttosto sensibili a tracce di ossigeno, e che i meccanismi di cattura sono piuttosto diversi.

È interessante domandarsi se con una miscela d'argon e di azoto sia possibile tenere in un vasto campo di tensioni applicate l'energia degli elettroni intorno ad 1 eV, cioè dove l'ossigeno ha piccole probabilità di cattura.

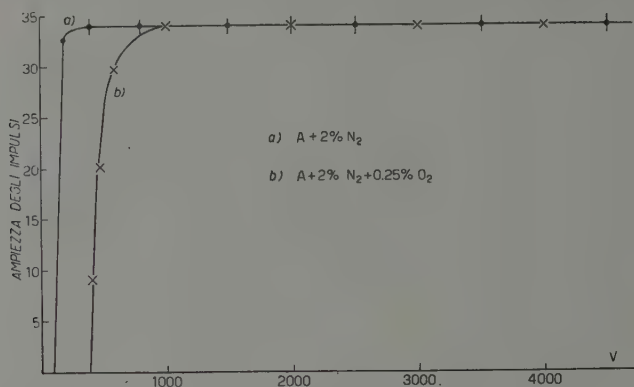


Fig. 5. Ampiezza di impulsi di particelle α in funzione della tensione applicata. Gas di riempimento: curva a) argon + 2% azoto; curva b) argon + 2% azoto + 0,25% ossigeno. Pressione: 200 cm Hg.

Fra le miscele studiate le più soddisfacenti da questo punto di vista si sono rivelate quelle con argon per il 98-96% e azoto per il 2-4%.

La fig. 5 mostra le curve di ampiezza degli impulsi ottenuti con la miscela argon + azoto (2%) (curva a) e con la stessa cui è stato aggiunto circa lo 0,25% di ossigeno. Dopo un lieve peggioramento della curva per bassi valori della tensione, questa quantità di ossigeno praticamente non altera la raccolta degli elettroni fino a 4000 V.

Aumentando le quantità di ossigeno le cose peggiorano: per esempio l'aggiunta dell'1% di ossigeno riduce ad un mezzo le ampiezze degli impulsi.

3. - La piccola influenza che l'ossigeno ha nelle miscele argon + 2-4% azoto permette un uso molto rapido della camera di ionizzazione. Spesso è utile rilevare lo spettro energetico di particelle α emesse da un nuclide avente vita media di pochi minuti o semplicemente si desiderano confrontare parecchi campioni α emittenti in un breve tempo. Queste necessità comportano un metodo rapido di messa in opera della camera di ionizzazione. Una rapida introduzione del campione e messa in opera della camera non è possibile con gas come l'argon o l'azoto perchè è richiesto per il loro buon funzionamento un vuoto spinto nella camera ed un lungo tempo di purificazione dei gas.

Viceversa l'uso della miscela argon-azoto può essere fatto senza avere un buon vuoto nella camera e non richiede alcuna purificazione spinta, provvisto che i gas usati siano inizialmente abbastanza esenti da ossigeno.

Nella fig. 6 è schematizzata un'apparecchiatura usata nel nostro laboratorio in un grande numero di misure di spettri di particelle α . La bombola contiene la miscela argon + 2% azoto molto pura (99,9%)⁽⁷⁾. Messa nella camera il campione emettitore α , questa viene chiusa e vuotata rapidamente (per 30-40 secondi) con la pompa rotativa; dopo di ciò la miscela di gas è introdotta nella camera, che è pronta per l'uso.

Tutta l'operazione dura uno o due minuti. Le curve di ampiezza ottenute con questo procedimento sono del tipo di fig. 5 e, anche lasciando nella camera qualche cm di Hg di aria, le curve di ampiezza sono abbastanza buone (come in fig. 5, b).

In queste condizioni è stata studiata la linearità della camera rilevando lo spettro di ampiezza di impulsi di particelle α . La fig. 7 mostra i risultati ottenuti nel caso di uranio naturale: la larghezza di righe ottenuta (2-3%) è quella usualmente ottenibile da questi strumenti.

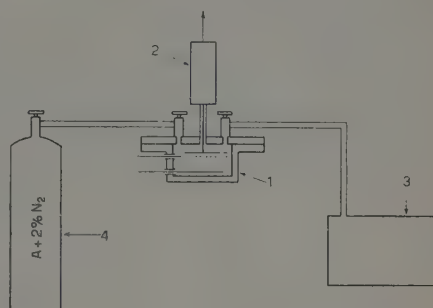
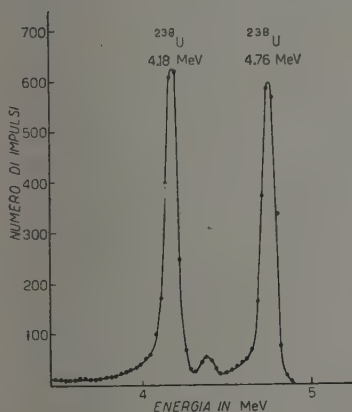


Fig. 6. Apparecchiatura per operazioni rapide. 1) Camera di ionizzazione. 2) Preamplificatore. 3) Pompa rotativa. 4) Bombola contenente la miscela argon+2% azoto.



Occorre infine dire che la miscela usata rimane di comportamento stabile per parecchi mesi senza nessuna cura speciale, nè purificazione.

Fig. 7. - Spettro di ampiezze degli impulsi di particelle α . Strato emittente: uranio naturale. Tensione totale: da 1000 a 4000 V. Il riempimento è fatto con l'apparecchiatura di fig. 6; la camera piena di aria è evacuata in 30 secondi con la pompa rotativa e quindi riempita con la miscela argon azoto. Il picco a 4.4 MeV è dovuto all' ^{235}U .

Gli autori ringraziano il prof. G. BOLLA per gli utili suggerimenti.

(7) La miscela viene acquistata dalla ditta S.I.O. (Milano) che la prepara correntemente.

SUMMARY

Gridded ionization chambers using electron collection, while very precise instruments for measuring the energy of α -particles, are not well suited to quick operation. The filling gas must be freed of even a trace of electron-capturing gas or vapor. This requires careful evacuation of the chamber and prolonged purification of the filling gas. In contrast, mixtures of A and 2-4% N_2 give good counting characteristics with up to 0.5% O_2 , much more O_2 than is found in commercially available supplies. No very special precautions need be taken. Thus α -particle spectra of short-lived emitters can be readily measured. Analysis of α -emitting nuclides contained in air is possible.

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Form Factors for Incoherent Scattering of High Energy Charged Particles by Nuclei.

V. BENZI

Istituto di Fisica dell'Università - Cagliari

(ricevuto il 18 Aprile 1955)

1. - Introduction.

Great interest has recently been devoted to the problem of evaluating the electromagnetic scattering of high energy electrons or muons by nuclei, in view of the numerous experiments carried out. In this paper we shall deal with the incoherent contribution to the cross sections for nuclei with $Z \leq 20$.

The incoherent contribution may be quite appreciable for light nuclei and at high energies, as shown by the calculations carried out by AMALDI, FIDECARO and MARIANI ⁽¹⁾. In the paper by AMALDI *et al.* the incoherent cross section is obtained by summing over all final states of the nucleus accessible to single particle excitation.

A closed general formula for the incoherent form factors has been derived by GATTO ⁽²⁾, for the case of incident particles of relatively high energy. GATTO has applied this formula to a statistical nuclear model (so as to get an approximate expression exhibiting the principal Z -dependence), and has moreover considered a more refined nuclear shell model, but treating only the case of ^{12}C in detail.

In this paper we shall extend these calculations to the other nuclei with $Z \leq 20$, so as to get simple closed expressions for their form factors for incoherent scattering.

The validity of our formulae will still be limited to the case of relatively energetic incident particles. A full discussion on this point will be found in the quoted paper by GATTO.

2. - Notations.

The incoherent form factor \mathcal{J}_Z is defined by

$$\left(\frac{d\sigma}{d\omega} \right)_{\text{inc.}} = \frac{1}{4} \frac{e^4}{c^4} \frac{F_0^2 Z^2}{p_0^2 \sin^4(\theta/2)} \mathcal{J}_Z,$$

⁽¹⁾ E. AMALDI, G. FIDECARO and F. MARIANI: *Nuovo Cimento*, **7**, 553 (1950).

⁽²⁾ R. GATTO: *Nuovo Cimento*, **10**, 1559 (1953).

where E_0 is the energy of the incident particle and p_0 the momentum in the laboratory system, which practically coincides with the center of mass system. For \mathcal{J}_z GATTO derives the expression ⁽²⁾

$$(1) \quad \mathcal{J}_z = Z^{-2} \left\{ Z - \sum_1^Z \sum_1^Z |\psi_i^*| \exp(i\mathbf{k} \cdot \mathbf{R}) |\psi_k|^2 \right\} = Z^{-2} \{ Z - O(Z) \} \leq Z^{-1}.$$

In formula (1), \mathbf{k} is a vector of modulus $k = 2K_0 \sin(\theta/2)$, where K_0 is the modulus of the initial wave-vector, and the ψ_i 's are the single particle wave functions.

We assume the shell model developed by JENSEN and MAYER ⁽³⁾. Therefore we suppose that a spin-orbit coupling is present. To avoid unnecessary complications, we take the potential well to be an oscillator potential $\frac{1}{2}m\omega r^2$, where m is the mass of the nucleon. Formula (1) does not account for the possible contribution due to the neutrons. An evaluation of this contribution is very difficult at present, for it results from mesonic processes; however, a simple phenomenological model leads to a very small prediction ⁽²⁾.

For all values of Z considered here, we may put $O(Z)$ in the form

$$(2) \quad O(Z) = \exp[-u](Z + a_z u^2 + b_z u^3 + c_z u^4)$$

with $u = (k\alpha)^2/2$ and $\alpha = 2R_0/\sqrt{2(2m+1)}$, where R_0 is the nuclear radius and m the total quantum number of the last proton. This estimate of α , which is the only parameter entering in this theory, is due to AMALDI *et al.* ⁽¹⁾, and it follows from the definition of the nuclear radius R_0 as the root of the mean square radius of the charge distribution.

3. - Table of the Coefficients.

We have calculated the coefficients a_z , b_z and c_z for every $Z \leq 20$, on the basis of the above model. Their values are indicated in Table I.

An approximate expression of \mathcal{J}_z is reported in a paper by COOPER and RAINWATER ⁽⁴⁾. If \mathcal{C}_z is the coherent scattering form factor, defined by

$$\left(\frac{d\sigma}{d\omega} \right)_{\text{coher.}} = \frac{1}{4} \frac{e^4}{c^4} \frac{E_0^2 Z^2 \cos^2(\theta/2)}{p_0^2 \sin^4(\theta/2)} \mathcal{C}_z,$$

then, following COOPER and RAINWATER,

$$(3) \quad \mathcal{J}_z \cong \mathcal{J}_z^* = Z^{-1}(1 - \mathcal{C}_z).$$

For an uniform charge density nuclear model, we have ⁽⁵⁾

$$\mathcal{C}_z = [3 (\sin kR_0 - kR_0 \cos kR_0) / (kR_0)^3]^2$$

where k and R_0 have the same meaning as before.

⁽²⁾ M. GOEPPERT MAYER: *Phys. Rev.*, **75**, 1969 (1949).

⁽⁴⁾ L. N. COOPER and J. RAINWATER: *Phys. Rev.*, **97**, 492 (1955).

⁽⁵⁾ L. I. SCHIFF: *Phys. Rev.*, **92**, 988 (1953).

TABLE I.

| | Z | a_z | b_z | c_z |
|----|-----|----------|----------|----------|
| Li | 3 | 1/4 | 0 | 0 |
| Be | 4 | 5/18 | 0 | 0 |
| B | 5 | 17/30 | 0 | 0 |
| C | 6 | 8/9 | 0 | 0 |
| N | 7 | 13/9 | 0 | 0 |
| O | 8 | 2 | 0 | 0 |
| F | 9 | 5/2 | 0 | 1/64 |
| Ne | 10 | 69/25 | 1/50 | 13/800 |
| Na | 11 | 73/25 | -1/50 | 1/32 |
| Mg | 12 | 81/25 | -1/50 | 29/800 |
| Al | 13 | 193/50 | -3/50 | 73/1600 |
| Si | 14 | 114/25 | -4/25 | 2/25 |
| P | 15 | 1216/225 | -131/225 | 157/900 |
| S | 16 | 1406/225 | -226/225 | 121/450 |
| Cl | 17 | 6581/900 | -599/450 | 499/1800 |
| Ar | 18 | 3509/450 | -322/225 | 109/360 |
| K | 19 | 7749/900 | -721/450 | 67/200 |
| Ca | 20 | 10 | -91/45 | 19/40 |

It is of interest to compare in some cases the values obtained by means of formula (3) with those which can be derived from our formulae (1) and (2). For values of $cp_0 \sim 200$ MeV we get, for instance

| | $\theta = 30^\circ$ | | $\theta = 60^\circ$ | | $\theta = 90^\circ$ | |
|--------------------|---------------------|---------|---------------------|---------|---------------------|---------|
| | J_z | J_z^* | J_z | J_z^* | J_z | J_z^* |
| ${}^7\text{Li}$ | 0.110 | 0.112 | 0.246 | 0.273 | 0.304 | 0.328 |
| ${}^{12}\text{C}$ | 0.073 | 0.075 | 0.144 | 0.155 | 0.162 | 0.166 |
| ${}^{27}\text{Al}$ | 0.033 | 0.050 | 0.063 | 0.076 | 0.074 | 0.076 |

Considering that the two models are sensibly different, we find that the agreement is in general satisfactory.

I would like to thank Dr. GATTO for useful discussions on the subject.

Note on the Properties of Two Functions Appearing in the Theory of TM Wave Propagation through Periodically Irisloaded Guides.

C. C. GROSJEAN

*Interuniversitair Instituut voor Kernwetenschappen,
Centrum van de Rijksuniversiteit Gent, Belgio*

(ricevuto il 22 Aprile 1955)

In a preceding article ⁽¹⁾, we have reexamined the problem of the propagation of circularly symmetric TM waves through infinite irisloaded wave guides. Starting with two unknown functions $g_1(x)$ and $g_2(x)$ in the expression for $E_z(a, z)$, we finally obtained the infinite system of equations (G 21) which we were able to transform into the equivalent system (G 24) and (G 26). Several properties of g_1 and g_2 have been discussed without solving the mentioned systems explicitly. In the present letter, we wish to complete our discussion on the following question.

We came to the conclusion that the possible dependence of g_1 and g_2 on the propagation constant β_0 can be expressed by

$$(1) \quad g_1(x, -\beta_0) = \pm g_1(x, \beta_0), \quad g_2(x, -\beta_0) = \mp g_2(x, \beta_0),$$

g_1 and g_2 being periodic functions of β_0 with period $2\pi/D$. But it was impossible to indicate at that time which signs are the correct ones if a certain pass band is considered. This can now be decided making some further calculations until we can apply some results obtained by VANHUYSE ⁽²⁾. Indeed, in the case of a plus sign in (1), the considered g -function will be cosine-like, whereas with a minus sign, the g -function will be sine-like. If they were expanded in Fourier series, one would get

$$(2) \quad - \text{ when } g(x, -\beta_0) = + g(x, \beta_0): \quad g(x, \beta_0) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos m\beta_0 D,$$

$$(3) \quad - \text{ when } g(x, -\beta_0) = - g(x, \beta_0): \quad g(x, \beta_0) = \sum_{m=1}^{\infty} b_m \sin m\beta_0 D.$$

⁽¹⁾ C. C. GROSJEAN: *Nuovo Cimento*, **1**, 427 (1955). References to formulae of this paper will be preceded by the symbol G.

⁽²⁾ V. J. VANHUYSE: *Nuovo Cimento*, **1**, 447 (1955).

Going over to zero or π modes, we see that in the case of a minus sign the corresponding g -function certainly vanishes. Let us now consider one of the mentioned infinite systems in (G), e.g. (G 26) and let us put $\beta_0 D$ respectively equal to 0 and π . We find

— for $\beta_0 D = 0$:

$$(4) \left\{ \begin{aligned} \frac{(-1)^{p-1} J_0(\xi_p b) G_1(p\pi)}{2\xi_p^2 a^2 J_0(\xi_p a) F_0(\xi_p; a, b)} &= \sum_{n=1}^{\infty} \frac{\sinh \frac{\kappa_n(D-d)}{2a}}{\sinh \frac{\kappa_n D}{2a}} \cdot \frac{G_1\left(\frac{i\kappa_n d}{2a}\right)}{\kappa_n^2 + \frac{4p^2 \pi^2 a^2}{d^2}} & (p = 0, 1, \dots) \\ \frac{(-1)^{q-1} J_0(\xi_q b) G_2[(q + \frac{1}{2})\pi]}{2\xi_q^2 a^2 J_0(\xi_q a) F_0(\xi_q; a, b)} &= \frac{1}{i} \sum_{n=1}^{\infty} \frac{\cosh \frac{\kappa_n(D-d)}{2a}}{\sinh \frac{\kappa_n D}{2a}} \cdot \frac{G_2\left(\frac{i\kappa_n d}{2a}\right)}{\kappa_n^2 + \frac{(2q+1)^2 \pi^2 a^2}{d^2}} & (q = 0, 1, \dots) \end{aligned} \right.$$

— for $\beta_0 D = \pi$:

$$(5) \left\{ \begin{aligned} \frac{(-1)^{p-1} J_0(\xi_p b) G_1(p\pi)}{2\xi_p^2 a^2 J_0(\xi_p a) F_0(\xi_p; a, b)} &= \sum_{n=1}^{\infty} \frac{\cosh \frac{\kappa_n(D-d)}{2a}}{\cosh \frac{\kappa_n D}{2a}} \cdot \frac{G_1\left(\frac{i\kappa_n d}{2a}\right)}{\kappa_n^2 + \frac{4p^2 \pi^2 a^2}{d^2}} & (p = 0, 1, \dots) \\ \frac{(-1)^{q-1} J_0(\xi_q b) G_2[(q + \frac{1}{2})\pi]}{2\xi_q^2 a^2 J_0(\xi_q a) F_0(\xi_q; a, b)} &= \frac{1}{i} \sum_{n=1}^{\infty} \frac{\sinh \frac{\kappa_n(D-d)}{2a}}{\cosh \frac{\kappa_n D}{2a}} \cdot \frac{G_2\left(\frac{i\kappa_n d}{2a}\right)}{\kappa_n^2 + \frac{(2q+1)^2 \pi^2 a^2}{d^2}} & (q = 0, 1, \dots) \end{aligned} \right.$$

In both cases, the system splits into two separate sets of equations having each their own frequency equation defining a number of resonance frequencies. In order to find to which pass band they belong, we can let d approach D . This continuous variation of d will cause a continuous shift of all the resonance frequencies towards those obtained for $d=D$, and it is reasonable to admit that no discontinuities will occur. The limiting case $d=D$ was examined by VANHUYSE ⁽²⁾, who found that the first set in (4) becomes simple (due to the vanishing right hand sides) and yields the zero modes of the odd pass bands, whereas the second set remains more complicated and gives rise to the remaining modes in the even bands. On the contrary, in the case of (5), the second set is easy to solve and yields the π modes in the even bands, whereas the first set remains complicated and gives rise to the other π modes in the odd pass bands. Therefore, we see that every 0 mode as well as every π mode in the odd pass bands of each (β_0, k) diagram is generated at a frequency satisfying the frequency equation of the first partial systems in (4) and (5). This leads consequently to a non-zero solution for $G_1(x)$ whereas the remaining partial system can only be satisfied by the trivial solution $G_2(x) \equiv 0$. Hence, in every band characterized by an odd number, only the function $g_1(x)$ enters the description of $E_z(a, z)$ in the

case of 0 or π modes, proving that *in the odd bands*, we must have

$$(6) \quad g_1(x, -\beta_0) = + g_1(x, \beta_0) ; \quad g_2(x, -\beta_0) = - g_2(x, \beta_0) .$$

In the same way, it can easily be shown that *in the even bands*, we must have the relations with the permuted signs:

$$(7) \quad g_1(x, -\beta_0) = - g_1(x, \beta_0) ; \quad g_2(x, -\beta_0) = + g_2(x, \beta_0) .$$

This should be taken into account, if β_0 -dependent trial functions for g_1 and g_2 were introduced in the theory, in the hope to find an approximate formula for the frequency curve $k(\beta_0)$ in each pass band.

Evidence for the K_{μ} -Meson.

J. K. BØGGILD, J. E. HOOPER, W. C. G. ORTEL and M. SCHARFF

Universitetets Institut for Teoretisk Fysik - Copenhagen (Denmark)

(ricevuto il 27 Aprile 1955)

Of 18 K-mesons found in half a stack of stripped emulsions flown by the 1953 Sardinia expedition, five are remarkably similar to the « K_{μ} -events» observed by the Pic-du-Midi group in their double cloud-chamber ⁽¹⁾.

While all the secondary particles leave the stack, six of them provide sections of track, between 2.2 cm and 7.9 cm long, on which measurements of multiple scattering and grain-density can be made. The emission energies of five of these are closely grouped, with a mean $p\beta c = (226 \pm 7)$ MeV. Since the residuals are in good accord with the known experimental errors, we believe that all the emission energies are, in fact, equal. The grain-density of these tracks was compared with that of known π - and μ -mesons of similar energy. It is clear that these secondaries are L-mesons, but closer identification is not possible by this method. That they are not electrons is shown by the fact that no evidence for energy loss by «*bremsstrahlung*» was found along 24 cm of track, corresponding in the emulsion to 8 radiation lengths. The secondary

of the sixth example is identified as a π -meson of initial $p\beta c = (170 \pm 8)$ MeV.

In three of the five examples, the primary is particularly long and flat, so that good measurements of the mass can be obtained by the mean gap-length vs. range method. The mean of the three is $(981 \pm 23) m_e$. Individual values are listed in Table I, together with other relevant data. As a check for systematic error, a τ -meson in the same stack was measured using exactly the same procedure. The measured mass, $(973 \pm 52) m_e$; is in good accord with the accepted value.

Knowing the mass of the decaying particle and the energy of emission of the charged secondary, one has strong evidence that the secondary is, in fact, a μ -meson. To produce a π -meson with the observed value of $p\beta c$, the decaying particle would have to be of mass $\gtrsim (1075 \pm 20) m_e$. Such a high energy π -meson could only be consistent with the measured value of the primary mass if the event were due not to decay, but to nuclear absorption, of the primary. Due to the internal motion of the nuclei in the absorbing nucleus, the secondary of such a process would not be emitted with a unique energy. The expected energy spread is about 50 MeV, appreciably greater than that observed.

(¹) B. GREGORY, A. LAGARRIGUE, L. LE-PRINCE-RINGUET, F. MULLER and CH. PEYROU: *Nuovo Cimento*, **11**, 292 (1954).

TABLE I.

| Event | Primary Mass (m_e) mean gap-length vs. range | $p\beta c$ (MeV) at decay | (*) | g (*) at decay |
|-------|--|------------------------------|-----|---------------------|
| 3 | — | 222 ± 13 | 15 | 0.95 ± 0.01 |
| 7 | 998 ± 35 | 228 ± 18 | 19 | 1.02 ± 0.02 |
| 12 | 969 ± 33 | 218 ± 13 | 15 | 0.94 ± 0.01 |
| 14 | — | 223 ± 13 | 15 | 0.97 ± 0.02 |
| 20 | 968 ± 60 | 230 ± 8 | 11 | 0.95 ± 0.01 |
| 5 | 996 ± 34 | 170 ± 6 | 8 | 1.12 ± 0.01 |

(*) This column gives the errors on $p\beta c$ when account is taken of a 3 % uncertainty in the scattering constant. All errors quoted are standard deviations.

Assuming the decay process postulated by GREGORY *et al.* (1), $K \rightarrow \mu + \nu$, one obtains $(1004 \pm 20) m_e$ for the primary mass, in good agreement with that obtained by direct measurement. Our five events thus provide further evidence of this decay scheme.

A more detailed account of this work will be published in the Communications of the Danish Academy of Sciences (Dan. Mat. Fys. Medd.).

Our sincere thanks are due Professor NIELS BOHR who has put the facilities of the Institute for theoretical Physics at our disposal and thus has enabled us to carry out this work. One of us (J.E.H.) is indebted to the Churchill and Rask-Ørsted Foundations, another (W.C.G.O.) gratefully acknowledges financial support received from the U. S. National Science Foundation during the course of this work.

On the Zenithal Effect of Extensive Air Showers (*).

D. BRINI, O. RIMONDI and U. VALDRÈ

Istituto di Fisica dell'Università - Bologna

I. FILOSOFO

*Istituto di Fisica dell'Università - Padova (**)*

(ricevuto il 28 Aprile 1955)

For low altitudes, the study of the zenithal dependence of extensive showers is essentially the study of an absorption phenomenon. The zenithal dependence function is then of the type:

$$(1) \quad f(x, \vartheta) = \exp \left[-\frac{x}{\lambda} \left(\frac{1}{\cos \vartheta} - 1 \right) \right],$$

where x is the atmospheric depth measured from the highest limit of the atmosphere.

For sufficiently small zenithal angles ϑ the (1) can be expressed approximately in the most convenient form as:

$$f(x, \vartheta) = \cos^n \vartheta \quad \text{with} \quad n = x/\lambda.$$

which is the one assumed by all Authors.

All the calculations and measurements made by the various Authors aim at establishing the exponent n of the cosine. There is a great spread in the values found in the literature and established by different methods: in particular, sea-level determinations are lacking (¹⁻⁹). It is, moreover, interesting to verify if there

(*) To this experiment the Padua Section and Bologna Group of the I.N.F.N. have also made a financial contribution.

(**) Now at the University of Illinois, Urbana, U.S.A.

(¹) J. DAUDIN: *Journ. de Phys.*, **6**, 302 (1945).

(²) M. DEUTSCHMANN: *Zeits. Naturf.*, **2a**, 61 (1947).

(³) R. W. WILLIAMS: *Phys. Rev.*, **74**, 1689 (1948).

(⁴) W. W. BROWN and A. S. MCKAY: *Phys. Rev.*, **87**, 1034 (1949).

(⁵) M. CRESTI, A. LORIA and G. ZAGO: *Nuovo Cimento*, **10**, 779 (1953).

(⁶) H. L. KRAYBILL: *Phys. Rev.*, **77**, 410 (1950).

(⁷) P. BASSI, A. M. BIANCHI, D. CADORIN and C. MANDUCHI: *Nuovo Cimento*, **9**, 1037 (1952).

(⁸) H. L. KRAYBILL: *Phys. Rev.*, **93**, 1362 (1954).

(⁹) P. BASSI, G. CLARK and B. ROSSI: *Phys. Rev.*, **92**, 441 (1953).

really exists a contribution from the large angles ($\vartheta > 40^\circ$) to the showers as would seem to be the case as shown by the experimental measurements of DEUTSCHMANN, BROWN and MCKAY, BASSI *et al.* and CRESTI *et al.* ^(2,4,5,9).

An experiment was, therefore, carried out at Bologna (60 metres above sea level), by means of a Geiger-counter device particularly sensitive to large angles.

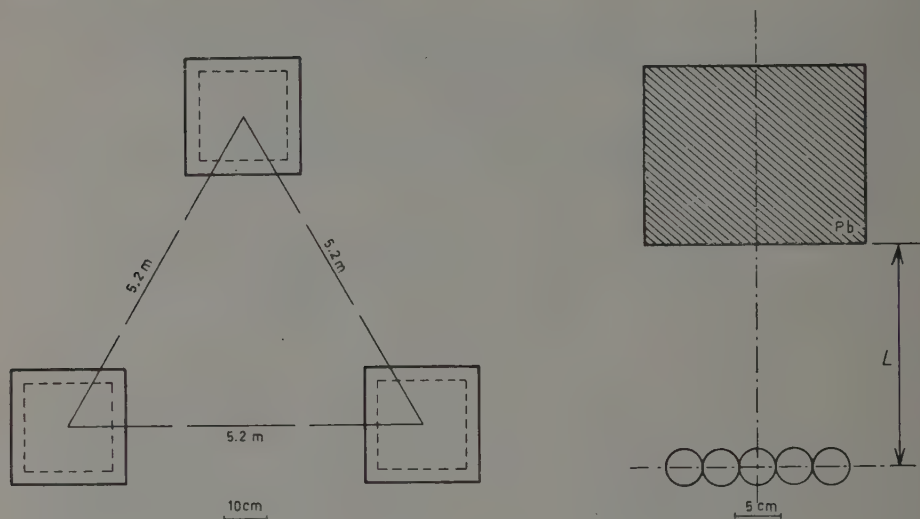


Fig. 1.

Fig. 1 illustrates the system used. Three counters trays, placed horizontally at the vertices of an equilateral triangle, show a different sensitivity to the showers coming from different zenithal angles when they are placed at different distances from a lead screen.

The useful surface of the trays was 400 cm². Measurements for values of $L = 50, 35, 25, 15$ and 10 cm were made.

The frequencies obtained were corrected from the background, by accidental coincidences and the barometric effect, assuming a coefficient $\alpha = 0,1 \text{ cm}^{-1} \text{ Hg}$ ⁽¹⁰⁾. The table reports the results obtained. The last column shows the ratio between the frequency at the distance $L = 50$ cm and that obtained for the other distances.

| L | Frequency (per h) | Ratio N_{50}/N_L |
|-----|-------------------|--------------------|
| 50 | 3.39 ± 0.11 | 1 |
| 35 | 2.57 ± 0.11 | 1.32 ± 0.10 |
| 25 | 1.84 ± 0.07 | 1.84 ± 0.13 |
| 15 | 0.94 ± 0.05 | 3.61 ± 0.20 |
| 10 | 0.47 ± 0.04 | 7.21 ± 0.77 |

The following hypotheses were formulated with the purpose to justify theore-

⁽¹⁰⁾ C. CASTAGNOLI, A. GIGLI and S. SCIUTI: *Nuovo Cimento*, **7**, 307 (1950).

tically the frequency measured by the device:

- 1) Primary radiation is isotropic;
- 2) There are not decay processes and the differential zenithal spectrum is of the type: $K A^{-\gamma} f(\vartheta) d\Delta d\Omega$;
- 3) The showers can be summarized as groups of particles with parallel trajectories.
- 4) The measurements of γ and K are independent of the zenithal angle and the density of the showers. The value 2.4 was taken for γ ⁽⁷⁾. We, moreover, obtained this value from a knowledge of the number of coincidences of the same order relative to counter trays with different surfaces. We got $\gamma = 2.43 \pm 0.10$.
- 5) The exponent n is independent of the density and the zenithal angle.

The first three hypotheses lead to the following expression for the threefold coincidences measured by the device with a distance L between the counter tray and lead screen:

$$(2) \quad N_L = \int_0^{2\pi} \int_{\vartheta_L(\varphi)}^{\pi/2} \int_0^{\infty} K A^{-\gamma} f(\vartheta) [1 - \exp[-A \cdot S_L(\vartheta, \varphi)]]^3 \sin \vartheta d\varphi d\vartheta dA.$$

$S_L(\vartheta, \varphi)$ is the projection of the sensitive area of a tray on a plane perpendicular to the direction of a shower and uncovered by the absorber. Its definition was made photographically and graphically.

With the position $A \cdot S_L(\vartheta, \varphi) = y$ (2) becomes through the hypothesis (4):

$$N_L = K f(\gamma) \int_0^{2\pi} d\varphi \int_{\vartheta_L(\varphi)}^{\pi/2} S_L(\vartheta, \varphi)^{\gamma-1} \cdot f(\vartheta) \sin \vartheta d\vartheta,$$

where

$$f(\gamma) = \int_0^{\infty} y^{-\gamma} (1 - \exp[-y])^3 dy,$$

Normalizing the counting rate to that relative to the distance $L_0 = 50$ cm we get:

$$(3) \quad \frac{N_{L_0}}{N_L} = F(L) = \frac{\int_0^{2\pi} d\varphi \int_{\vartheta_{L_0}(\varphi)}^{\pi/2} S_{L_0}(\vartheta, \varphi)^{\gamma-1} \cdot f(\vartheta) \cdot \sin \vartheta d\vartheta}{\int_0^{2\pi} d\varphi \int_{\vartheta_L(\varphi)}^{\pi/2} S_L(\vartheta, \varphi)^{\gamma-1} \cdot f(\vartheta) \cdot \sin \vartheta d\vartheta}.$$

We carried out the calculation of (3) numerically assuming, in the first place, $f(\vartheta) = \cos^n \vartheta + B$, where B is a constant. The introduction of B is suggested by

the fact that the device feels the effect of little showers produced at the edges of the screen and the scattering effect of the particles of a shower.

This computation, which leads to a value $n = 6$, was carried out to compare our results with those of other Authors. At sea level, BASSI and cow., with Geiger-counters, found $n = 3$, DEUTSCHMANN, with cloud chamber, $n = 8$, and KRAYBILL, with altitude dependence, $n = 7.5$.

With, however, the purpose of seeing the influence of large angles, for which the cosine at the exponential is no longer approximable, we sought agreement using:

$$f(\theta) = \frac{\exp[-A/\cos \theta] + B}{\exp[-A] + B},$$

which the minimum squares method gave in the:

$$(4) \quad f(\theta) = \frac{\exp[-S/\cos \theta] + 2 \cdot 10^{-4}}{\exp[-S] + 2 \cdot 10^{-4}}.$$

With that, the influence of the large angles at the zenithal effect is 20% at 40°, and 5% at 50°.

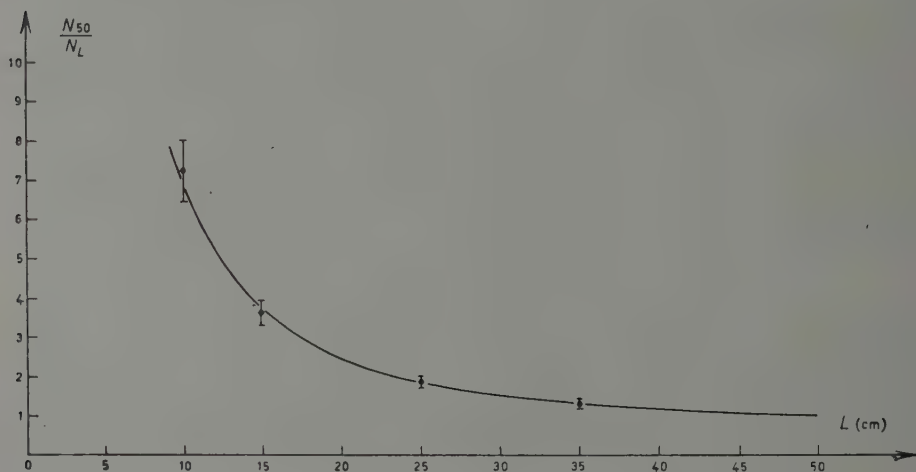


Fig. 2.

Fig. 2 reports the experimental points and the solid curve gives the behaviour of (3) assuming the expression (4) for $f(\theta)$.

It must be observed, however, that the exponents from $n \cong 4$ and $n \cong 7$ give curves which are still well suited to the experimental data. Taking all the possible values of the (4) exponent into account, we get some path lengths which are rather high in respect to that of primary radiation. That should not be surprising, since it is a consequence of the shower development lengths.

Our thanks are due to the Director of the Institute of Physics of the University of Bologna, Professor G. PUPPI, for his valuable advice, and Professor A. ROSTAGNI, Director of the Institute of Physics of the University of Padua for his collaboration.

Non-Linear Interactions and Nuclear Saturation.

E. CLEMENTEL and C. VILLI

Istituti di Fisica delle Università di Padova e di Trieste
Istituto Nazionale di Fisica Nucleare - Sezione di Padova

(ricevuto il 5 Maggio 1955)

In a previous note ⁽¹⁾ it was shown that the many body treatment of the heavy nucleus in the Wigner forces approximation is formally equivalent to the introduction of non-linear interactions, provided the non-linearities are expressed as an uniformly convergent series of the meson field amplitude, i.e. $F[\varphi] = \sum_{n=2} a_n \varphi^n$. Under this assumption the interactions between nucleons are described as arising from mesons which obey the non-linear, classic, scalar and time-independent wave equation:

$$(1) \quad -\Delta\varphi(\mathbf{r}) + \mu^2\varphi(\mathbf{r}) + F[\varphi(\mathbf{r})] = g\rho(\mathbf{r}),$$

where $(\hbar = c = 1) \mu^{-1} = 1.4 \cdot 10^{-13}$ cm is the meson Compton wave length, g the meson-nucleon coupling constant and $\rho(\mathbf{r})$ the nucleon density. Many body forces of a complex nature have been also derived by inspection of the mathematical properties of the Schiff-Thirring equation ⁽²⁾, but the customary treatment of the saturation of heavy nuclei on the basis of the φ^3 non-linearity, either in the scalar ⁽³⁾ or in the neutral pseudoscalar ⁽⁴⁾ case, cannot be reconciled with many body forces, which have been shown to lead to saturation assuming, somewhat arbitrarily, an enormous core effect on the nucleon kinetic energy ⁽⁵⁾. So far, however, only the φ^3 and the Born-Heisenberg non-linearity ⁽⁶⁾ have been considered in these problems. The former, apart from the unreliable values of the coupling constant g and the non-linearity strength λ obtained to fulfill the saturation prescription of the binding energy, is conflicting both with multiple production of mesons ⁽⁷⁾ and the scattering

⁽¹⁾ E. CLEMENTEL and C. VILLI: *Nuovo Cimento*, **1**, 344 (1955).

⁽²⁾ R. O. FORNAGUERA: *Nuovo Cimento*, **1**, 132 (1955).

⁽³⁾ B. MALENKA: *Phys. Rev.*, **86**, 68 (1952); P. MITTELSTAEDT: *Zeits. Phys.*, **137**, 545 (1954).

⁽⁴⁾ E. M. HENLEY: *Phys. Rev.*, **87**, 42 (1952).

⁽⁵⁾ S. D. DRELL and K. HUANG: *Phys. Rev.*, **91**, 1527 (1953).

⁽⁶⁾ W. HEISENBERG: *Zeits. Phys.*, **113**, 61 (1939); **126**, 519 (1949); **133**, 65 (1952).

⁽⁷⁾ W. HEISENBERG: *Komische Strahlung* (Berlin, 1953), p. 151.

of pions from nuclei ⁽⁸⁾, whereas the latter is inadequate to explain the saturation of nuclear forces ⁽⁹⁾. Although the situation is clearly disconcerting, we shall try to develop some consequences of Eq. (1) with regard to heavy nuclei.

To reduce the number of unknowns, we write Eq. (1) in the form:

$$(2) \quad -\Delta\varphi(\mathbf{r}) + \mu^2\Phi[\varphi(\mathbf{r})]\varphi(\mathbf{r}) = g\varrho(\mathbf{r}),$$

where $\Phi[\varphi] = \mu^{-2} \sum_0^\infty a_n \varphi^n$ ($a_0 = \mu^2$). We now assume that $\mu^{-2}a_n = \lambda^n$, λ being the non-linearity strength to be determined from experiments as well as the coupling constant g . Since the convergence of $\Phi[\varphi]$ for all points of the domain where the meson field is defined requires $\lambda\varphi < 1$, the series $\Phi[\varphi]$ can be summed and Eq. (2) reads

$$(3) \quad -\Delta\varphi(\mathbf{r}) + \mu^2\varphi(\mathbf{r})[1 - \lambda\varphi(\mathbf{r})]^{-1} = g\varrho(\mathbf{r}).$$

The Hamiltonian associated with this equation is given by

$$(4) \quad H_A = \int \{ (1/2)(\nabla\varphi)^2 - (\mu/\lambda)^2 \log(1 - \lambda\varphi) - (\mu^2/\lambda)[1 + (\lambda g\varrho/\mu^2)]\varphi \} d\mathbf{r}.$$

At this stage we assume as a zero approximation that $\varphi(\mathbf{r})$ varies slowly in the interior of the nucleus having uniform density. Taking $\varphi(\mathbf{r}) = \varphi_0$ and $\varrho(\mathbf{r}) = \varrho_0$ for $r < R$, the zero approximation of the meson field amplitude is

$$(5) \quad \varphi_0 = (g\varrho_0/\mu^2)[1 + (\lambda g\varrho_0/\mu^2)]^{-1}.$$

The field obeying Eq. (2), for a given value of g , is a function of the nuclear density and a monotonically decreasing function of the parameter λ . Its limit, as in the φ^3 case, is zero for R or λ tending to infinity. An essential difference, however, comes out when the limiting behavior of the field is examined by increasing g indefinitely. In the φ^3 case, for a given nuclear radius R , the field amplitude $\varphi_0 \sim g^{-1}$ tends to zero as g approaches to infinity, while in our case φ_0 tends to λ^{-1} and in the linear case ($\lambda=0$) the field, everywhere proportional to g , tends to infinity. It follows that the limiting behavior of the meson field as a function of the coupling constant is not disentangled by the strength of the non-linearity, but appears to be ruled by it. Whether non-linearities have any physical meaning, this mathematical circumstance must imply some physical feature of the saturation mechanism, much more essential than the modification of the value of the coupling constant required by the φ^3 non-linearity. Clearly, as g increases and φ_0 tends to λ^{-1} , the convergence of $\Phi[\varphi]$ gets worse, unless the nuclear density decreases. It follows that the convergence requirements, on which Eq. (3) is based, will make the values of g , λ and R , adequate to account for saturation, rather critic.

The potential energy per nucleon, following from Eq. (4) and (5), reads

$$(6) \quad H_A/A = -\alpha[1 - \beta^{-1} \log(1 + \beta)] = -\alpha f_0(\beta),$$

where $\alpha = g/\lambda$ and $\beta = \lambda g\varrho_0/\mu^2$. For $\beta < 1$, Eq. (6) reduces, with some obvious

⁽⁸⁾ W. THIRRING: *Zeits. für Naturf.*, **9**, 804 (1954).

⁽⁹⁾ F. CAP: *Phys. Rev.*, **95**, 287 (1954).

change of notations, to Eq. (7) of the previous note ⁽¹⁾. Let $T_A^* = \gamma T_A$ denote the total kinetic energy of the nucleons, where the factor $\gamma \geq 1$ takes account of core effects and T_A is given by the Thomas-Fermi model of the nucleus. Then, from the condition $\partial W_A / \partial R = 0$, W_A being the total energy of the nucleus, one gets

$$(7) \quad (\alpha/\gamma)^{\frac{1}{2}} = (\xi/r_0) f_1(\beta),$$

where

$$(8) \quad \xi = (3\pi/20M)^{\frac{1}{2}}(3/\pi)^{\frac{1}{2}}, \quad f_1(\beta) = [\lambda\varphi_0 - f_0(\beta)]^{-\frac{1}{2}}.$$

Eq. (7) then becomes

$$(9) \quad H_A/A = -\gamma(\xi/r_0)^2 f_0(\beta) f_1^2(\beta).$$

| β | 0 | 0.5 | 1.0 | 2.0 | 3.0 | 3.5 | 5.0 | ∞ |
|--------------|----------|-------|-------|-------|-------|-------|-------|----------|
| $f_0(\beta)$ | 0 | 0.189 | 0.307 | 0.451 | 0.538 | 0.570 | 0.642 | 1 |
| $f_1(\beta)$ | ∞ | 2.659 | 2.276 | 2.152 | 2.172 | 2.195 | 2.284 | ∞ |

Eq. (9), written explicitly as a function of g , β , r_0 , is

$$(10) \quad H_A/A = -(3/4\pi r_0^3)(g/\mu)^2[(1+\beta)^{-1} - (\gamma\xi/\alpha\beta r_0^2)].$$

Eqs. (7) and (10), for a certain value $\beta = \beta_0$, must be consistent with experimental values of r_0 and the nuclear binding energy per nucleon. Since we neglect Coulomb effects, we assume as a binding energy 95% of the volume term in the empirical binding energy formula ⁽¹⁰⁾. Ignoring core effects ($\gamma=1$) and assuming $r_0 = \mu^{-1}$, Eqs. (7) and (10) are satisfied provided $\beta_0 = 3.5$. It follows that $\alpha = 49.1$ and $g^2 = (4\pi\alpha\beta/3)r_0 = 5.1$, whereas the φ^3 non-linearity requires $g^2 \sim 40$. Furthermore, it is found $\lambda\varphi_0 = 0.777$ and the potential in which a nucleon moves within the nucleus is $V_0 = -g\varphi_0 = -\alpha\beta(1+\beta)^{-1} = -38.2$ MeV, a value not much different from the one calculated in the φ^3 case ($V = -34.2$ MeV). Finally, it is seen that, according to Eq. (3), the non-linearity strength λ , insuring the saturation, is very weak, i.e. 1/20 of the value obtained with the φ^3 term only.

To take this investigation a step further, we evaluate the meson field to the first approximation considering still a heavy nucleus in its ground state. Assuming $\varphi_1 \ll \varphi_0$, Eq. (3) reads

$$(11) \quad -\Delta\varphi(\mathbf{r}) + \mu^2\Phi[\varphi_0]\varphi(\mathbf{r}) = g\varrho(\mathbf{r}),$$

which appears as a linear equation describing mesons having "effective mass"

$$(12) \quad \mu^* = \mu\Phi^{\frac{1}{2}}[\varphi_0] = \mu[1 + (\lambda g\varrho_0/\mu^2)]^{\frac{1}{2}}.$$

⁽¹⁰⁾ E. FERMI: *Nuclear Physics* (University of Chicago Press, 1950), p. 7.

According to Eq. (12) it is seen that the « effective mass » coincides with the pion mass only if the nuclear density is zero or there are no interactions or no non-linearities. With the values of g and λ previously determined the effective mass comes out to be about 636 electron masses.

Although a closer inspection of the behavior of the meson field obeying Eq. (3) shows that the effective mass is probably not the consequence of a mathematical device as, for instance, the electron effective mass in the band approximation, we shall not discuss in terms of these oversimplified arguments the possibility of a physical reality of such mesons. However, if they were to be produced, the production yield could not have, neglecting absorption, an $A^{\frac{2}{3}}$ -dependence. For the moment, we regard Eq. (12) as an unexpected clue to understand the behavior of the nuclear forces mesons when collective interactions are operating, as in the case of heavy nuclei.

It is interesting to note that no effective masses smaller than the pion mass are allowed by this theory. In fact, only if $\Phi[\varphi] = (1 + \lambda\varphi)^{-1}$ we would have $\mu^* < \mu$, but in this case the convergence of the series $\Phi[\varphi]$ would require $\beta < 0.5$, and it can be shown that this condition excludes saturation.

To find the spatial dependence of the nuclear potential we assume spherical symmetry and solve the equations

$$(14a) \quad -\Delta\varphi(\mathbf{r}) + \mu^{*2}\varphi(\mathbf{r}) = g\varrho_0, \quad (r < R),$$

$$(14b) \quad -\Delta\varphi(\mathbf{r}) + \mu^2\varphi(\mathbf{r}) = 0, \quad (r > R).$$

It is found

$$(15a) \quad V_{\text{int}}(r) = -(g^2\varrho_0/\mu^{*2}) \left\{ 1 - \frac{(R/r)(1 + 1/R\mu)(\sinh(\mu^*r))}{\sinh(\mu^*R) + (\mu^*/\mu)\cosh(\mu^*R)} \right\} \quad (r < R),$$

$$(15b) \quad V_{\text{ext}}(r) = -(g^2\varrho_0/\mu^{*2}) \frac{\exp(\mu R)[(\mu^*/\mu) - (1/R\mu^*)\operatorname{tgh}(\mu^*R)] \exp[-\mu r]}{(\mu^*/\mu) + \operatorname{tgh}(\mu^*R)} \frac{1}{r} \quad (r > R).$$

The effective potential in which a nucleon moves is shaped somewhat between the square well and the isotropic oscillator potential well. For $r \ll R$ we obtain for $V_{\text{int}}(r)$ the value V_0 previously determined.

Some implications of the linearization of Eq. (3) by means of the introduction of an effective meson mass will be considered in a forthcoming paper.

We thank Profs. N. DALLAPORTA and W. THIRRING for some discussions.

Saddle Shapes, Threshold Energies and Fission Asymmetry on the Liquid Drop Model.

U. L. BUSINARO

Laboratori CISE - Milano

S. GALLONE

Istituto di Fisica dell'Università e Laboratori CISE - Milano

(ricevuto il 6 Maggio 1955)

In a recent paper by the same authors ⁽¹⁾ a formula was given for the deformation energy of the nuclear drop. This formula can be applied to nuclear shapes which are not too different from a prolate ellipsoid of revolution.

In the system of coordinates defined in B.G. (6) the droplet shape is described by a Legendre expansion of the following type,

$$(1) \quad r(\mu) = a \left[1 + \sum_{i=1}^{\infty} a_i P_i(\mu) \right].$$

In the previous paper attention was paid above all to the effect on the nuclear energy of a P_3 deformation superimposed on the basic ellipsoid. The conclusion was reached that such a P_3 deformation could lower the potential energy of the system provided the eccentricity of the basic ellipsoid were sufficiently large. The effect of the P_2 deformation on the fission energy is however certainly not negligible for nuclei with $x < 0.9$. In the present paper therefore the influence of the P_2 deformations is examined in detail. The additional coefficients of B.G. (8) (see also B.G. Appendix IV) (*) necessary to the numerical analysis of the contribution of

⁽¹⁾ U. L. BUSINARO and S. GALLONE: *Nuovo Cimento*, **1**, 629 (1955), quoted here as B.G.

(*) The coefficient \bar{C}_{is} of B.G. (Appendix IV) must be corrected as follows:

$$\bar{C}_{is} = \frac{1}{y^{i+s}} \frac{1}{(1-y^2)^{\frac{s}{2}}} \left[M_{is}(y) + N_{is}(y) \frac{1}{\sqrt{1-y^2}} \frac{\arcsin y}{y} \right].$$

these deformations are:

$$(2) \quad \left\{ \begin{aligned} \bar{B}_2 &= \frac{1}{240y^2(1-y^2)^{\frac{3}{2}}} \cdot \left[(-225 + 210y^2 + 32y^4) + (225 - 360y^2 + 152y^4) \frac{(\arcsin y)/y}{\sqrt{1-y^2}} \right] \\ B_2 &= \frac{1}{42y^2(1-y^2)^{\frac{3}{2}}} \left[(63 - 33y^2) + (-63 + 54y^2 - 11y^4) \frac{Q_2(1/y)}{y} \right] \end{aligned} \right.$$

$$(3) \quad \left\{ \begin{aligned} R_{22} &= \frac{1}{1260} (645y^8 - 4905y^4 + 10215y^2 - 6615) \\ S_{22} &= \frac{1}{1260} (-17y^8 - 1440y^6 + 7470y^4 - 12420y^2 + 6615) \\ M_{22} &= \frac{1}{5760} (256y^8 + 1236y^6 - 11412y^4 + 25245y^2 - 15390) \\ N_{22} &= \frac{1}{5760} (1744y^8 - 10872y^6 + 29178y^4 - 35505y^2 + 15390) \end{aligned} \right.$$

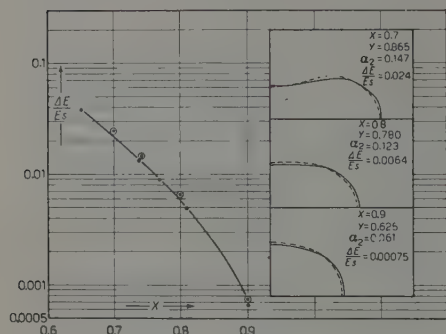


Fig. 1. — Comparison between FRANKEL and METROPOLIS (2) saddle shapes (dotted line) and those here obtained by approximate method and defined by indicated values of parameters. Threshold energies obtained in this work (circles) are compared with results of electronic machine computation (2) (dots) on semi-logarithmic plot.

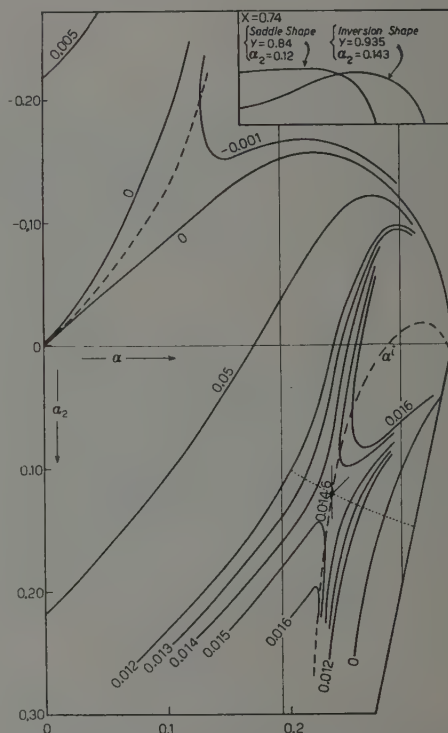


Fig. 2. — Contour plot of the energy surface $\Delta E/E_s$ drawn in the plane (α, α_2) . Dashed line is given by equation $\partial(\Delta E/E_s)/\partial\alpha_2 = 0$. Thresholds reported in fig. 2 were computed by finding minimum energy value along similar lines. The «inversion line» $\alpha = \alpha^i$ is also shown. Consistently with the adopted approximation, separation into distinct fragments takes place along curved edge at the lower right hand corner of the figure. The saddle shape and the shape corresponding to the «inversion point» on a tentative «reaction path» (dotted line) are shown.

Using these coefficients a contour plot of the energy surface may be drawn in the (α, α_2) plane⁽⁺⁾. Such a plot is given in Fig. 2 for $x = 0.74$. The map shows a saddle point whose energy ($AE/E_s = 0.0146$) may be compared with that given for the same x value by FRANKEL and METROPOLIS⁽²⁾ i.e. 0.0136. The agreement with FRANKEL and METROPOLIS electronic machine computations is illustrated in Fig. 1 where threshold energies and saddle shapes are compared for various x values.

Fig. 2 shows the «inversion line» $\alpha = \alpha^i$ which indicates the beginning of instability with respect to P_3 deformations (as already discussed in B.G.); a possible «reaction path» is also shown together with the shape corresponding to the «inversion point» which would be reached along this path.

In Fig. 3 are compared (as in B.G. Fig. 3) the eccentricities \bar{y} of the saddle shapes and the inversion eccentricities y^i as a function of x . The intersection of the two curves would correspond to the appearance of asymmetric saddles. However nothing definite can be deduced as to the existence and location of such asymmetric saddles, due to the approximations of the calculations.

The results given in this paper seem to confirm the conclusion of HILL and WHEELER⁽³⁾ that the liquid drop model may provide a satisfactory interpretation of asymmetric fission.

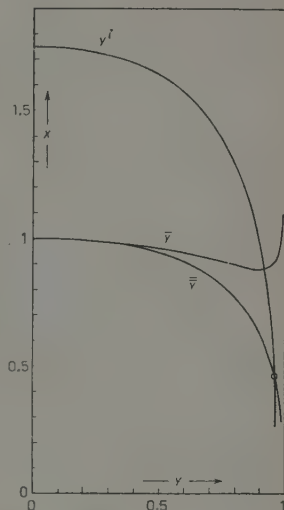


Fig. 3. — Similar to B.G. fig. 3. Curve \bar{y} gives the saddle shape eccentricity, as function of Bohr parameter x when P_2 contribution is taken into account. Curve y^i gives the «inversion eccentricity», while \bar{y} corresponds to «saddle shape» eccentricity for purely ellipsoidal deformations. The intersection of y^i with \bar{y} should mean appearance of asymmetric saddle shapes. However the adopted approximation probably fails in this region.

(⁺) α is a parameter defined in B.G. (13):

$$\alpha = y^2/3,$$

where y is the eccentricity of the basic ellipsoid.

(²) S. FRANKEL and J. K. METROPOLIS: *Phys. Rev.*, **72**, 914 (1947).

(³) D. L. HILL and J. A. WHEELER: *Phys. Rev.*, **89**, 1102 (1953).

Au sujet du volume sensible des compteurs de Geiger-Müller à cathode externe.

D. BLANC

Laboratoire de Physique Atomique et Moléculaire du Collège de France - Paris

(ricevuto il 7 Maggio 1955)

Dans une récente publication ⁽¹⁾, A. M. BAPTISTA, R. H. CORDEIRO et J. P. GALVAO ont étudié des compteurs à parois de verre et graphitage externe du type décrit par D. BLANC et M. SCHERER ⁽²⁾. Ce dispositif élimine toutes les impulsions parasites pouvant provenir des régions terminales (fig. 1). Ils ont montré qualitativement que le volume sensible du détecteur est d'autant plus grand que la longueur isolée entourant la zone graphitée placée à la masse est plus importante. Dans notre publication nous indiquons que la longueur efficace L coïncide avec la longueur $L_c = CD$ placée à la masse, et c'est pourquoi ces auteurs concluent que leurs résultats sont en désaccord avec nos expériences.

Ce désaccord apparent nécessite quelques précisions. Nos recherches ⁽²⁾ portaient sur des compteurs destinés à faire partie de plateaux de coïncidence et anti-coïncidence pour études du rayonnement cosmique: la longueur graphitée placée

à la masse était importante (de l'ordre de 60 cm). D'autre part, le verre « novo » utilisé pour fabriquer ces détecteurs possède une résistivité suffisamment élevée pour que la longueur isolée totale $L_i = BC + DE$ soit inférieure à 2 cm. L_i représentant environ 3 pour cent de L_c , il est légitime de considérer que L est égale à CD .

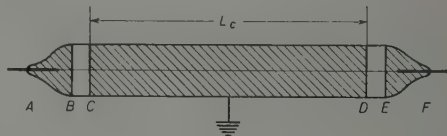


Fig. 1.

Par contre, l'utilisation récente de verres UVK-1 et UVK-2 plus conducteurs ⁽³⁾ nécessite des valeurs de L_i plus grandes: la différence entre L_c et L risque de ne plus être négligeable, ce qui m'a incité à effectuer les expériences complémentaires qui suivent. Les remplissages utilisés sont du type courant: éthanol sous la pression de 1 cm de mer-

⁽¹⁾ A. M. BAPTISTA, R. H. CORDEIRO et J. P. GALVAO: *Rev. da Fac. de Ciencias de Lisboa*, 4, 5 (1955).

⁽²⁾ D. BLANC et M. SCHERER: *C. R. Acad. Sciences*, 228, 2018 (1949).

⁽³⁾ D. BLANC: *Journ. de Phys. Rad.*, 15, 693 (1954).

cure et argon sous la pression de 9 cm de mercure.

1. — Pour mesurer L , j'ai utilisé des compteurs à cathode multiple du modèle représenté dans la fig. 2, chaque section

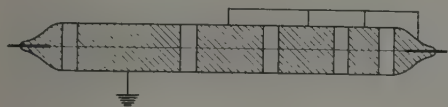


Fig. 2.

graphitée fonctionnant séparément si toutes les autres sont placées à la tension du fil. Ce dispositif équivaut à une série de compteurs de même diamètre, de remplissages identiques, donc de même efficacité, mais de longueur L_c variable. Le compteur étant placé dans un flux radioactif uniforme, le taux de comptage N_0 pour le seuil de Geiger porté en fonction de L_c varie linéairement, mais présente une certaine ordonnée à l'origine (fig. 3). L'intersection de cette droite

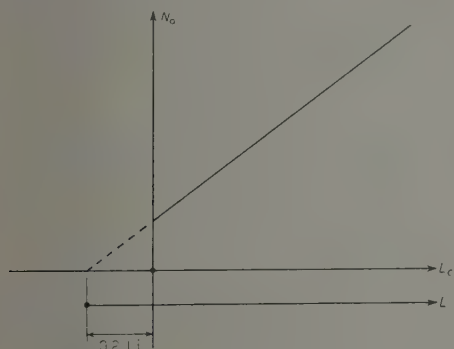


Fig. 3.

avec l'axe des abscisses donne la valeur exacte de L . Quelle que soit la nature du verre, pour des diamètres internes allant de 0,5 à 3 cm :

$$(1) \quad L = L_c + 0.2 L_i,$$

cette relation étant valable pour le seuil de Geiger.

2. — Les expériences qui précèdent ont été faites pour L_i inférieure ou égale à 4 cm. La vérification de la formule (1) pour des valeurs de L_i beaucoup plus grandes est réalisable à l'aide de compteurs du modèle représenté sur la fig. 4,

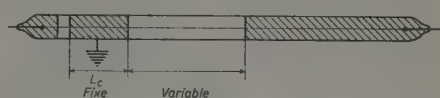


Fig. 4.

où la valeur de L_i peut être 3 ou 4 fois plus grande que L_c . La fig. 5 donne le taux de comptage N_0 en fonction de la longueur $(L_c + 0.2 L_i)$ pour un compteur de 3 cm de diamètre: la courbe obtenue est une droite passant par l'origine et la relation (1) est vérifiée pour des valeurs de L_i allant jusqu'à 40 cm.

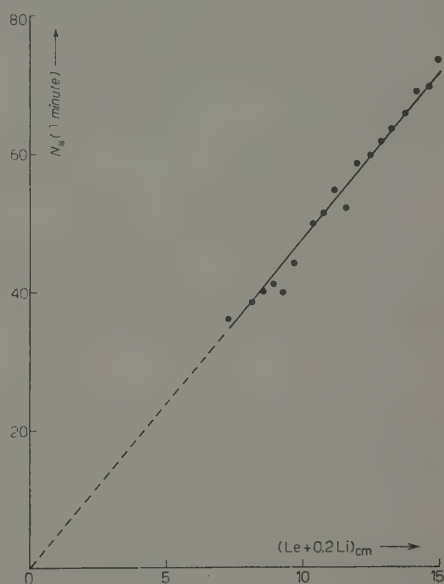


Fig. 5.

D'autre part, la pente du palier augmente avec L_i en même temps que sa longueur diminue, ce qui confirme les résultats obtenus antérieurement (2).

On the Nuclear Interaction of 350 MeV α -Particles.

G. QUARENI and G. T. ZORN (*)

Istituto di Fisica dell'Università - Padova
Istituto Nazionale di Fisica Nucleare - Sezione di Padova

(ricevuto il 12 Maggio 1955)

In a previous investigation the interactions in photographic emulsions of He-nuclei of the Cosmic Radiation were studied and the general characteristics of the nuclear disintegrations produced at primary energies greater than 650 MeV were determined ⁽¹⁾. In this investigation the study has been extended to the primary energy of 350 MeV.

The photographic emulsions employed in this investigation were Ilford type G5, 200 and 400 μ in thickness. They were exposed to the 385 MeV α -particle beam of the Berkeley cyclotron and were kindly made available to us by Prof. W. H. BARKAS.

a) Interaction Mean Free Path Determination.

In order to assure that all nuclear interactions and all « single » scatterings with projected angles greater than 3° were recorded, « along the track » scanning was employed. The events observed may be divided into four groups:

1) « single » scatterings where the angular deviation of the track did not exceed 20° ; 2) « single » scattering events where the deviation exceed 20° ; 3) sudden change in the grain density of the α -particle track; 4) stars composed of at least two charged secondaries. The first group has been considered separately and their angular distribution has been compared with that calculated using the Rutherford formula for Coulomb scattering. Within experimental error no divergences from this simple theory were found. We have therefore classified these events as due to single coulomb scatterings. The events in group 2 together with those in groups 3 and 4 have been classified as « nuclear » events, and were used in the determination of the mean free path for nuclear interaction.

In a total length of 2912.1 cm, 145 « nuclear » events were observed. This corresponds to a mean free path for nuclear interaction of 20.02 ± 1.67 cm or 80 ± 6.75 g/cm². Taking into account energy losses due to ionization, the average energy at which this determination was made was 350 MeV.

Previous estimates of the mean free path for nuclear interaction of the α -particles in photographic emulsions were made

(*) Now at Brookhaven National Lab. Upton, L. I. N. Y. (U.S.A.).

(1) M. CECCARELLI, G. QUARENI and G. T. ZORN: *Nuovo Cimento*, 1, 669 (1955).

by GARDNER⁽²⁾ at primary energies ranging from 50 to 250 MeV. He obtained values greater than twice that obtained in our investigation in the energy region of 350 MeV. The comparatively high value of the mean free path would not seem to have its explanation in a rapid energy dependence of the m.f.p.. Perhaps the discrepancies may be attributed to the low sensitivity of the emulsions used in that experiment.

Our results, however, seem to be in good agreement with those of WADDINGTON⁽³⁾ for «cosmic ray» α -particles as well as with the results obtained recently by the Berkeley group⁽⁴⁾ for α -particles of 385 MeV.

b) Analysis of the Nuclear Disintegrations Produced.

In this study we have chosen as a basis for comparison, the disintegrations produced by protons with the same energy per nucleon. In particular we have compared our results with those reported by HODGSON⁽⁵⁾ on disintegrat-

ions produced by 94 MeV proton stars in C2 emulsion. In this comparison we have normalized his data to compare with our results.

The distribution of α -star sizes, based on 306 events, is given in Table I and it will be seen that the average size and the half width of the distribution are greater than those for protons of 94 MeV. The numbers and percentages, reported in lines 4 and 5, correspond respectively to the stars with one or with two singly charged particles having the same velocity as the incident α -particles.

Since the emulsions were under-developed, it was possible to distinguish by ionization between slow doubly charged particles and slow singly charged particles. Basing our criteria on the effects due to the Coulomb barrier, we have separated the disintegrations of heavy nuclei from those of light nuclei. Such an analysis was made of 43 α -stars, and 10 of these were recognized to be produced by disintegrations of light elements. The remaining 33 events, which we have considered to represent disintegrations of heavy nuclei, were studied

TABLE I.

| | Number of prongs | | | | | | | | | Total of the stars |
|-----------------|------------------|------|------|------|------|------|-----|-----|-----|--------------------|
| | 1 | 2 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | |
| P-stars | 50 | 87 | 25 | 22 | 14 | 1 | | | | 198 |
| % | 25.2 | 43.9 | 12.6 | 11.1 | 6.6 | 0.5 | | | | 100% |
| α -stars | 10 | 80 | 58 | 49 | 53 | 31 | 18 | 5 | 2 | 306 |
| % | 3.3 | 26.1 | 18.9 | 16.0 | 17.3 | 10.1 | 5.9 | 1.6 | 0.6 | 100% |
| | 4 | 14 | 10 | 5 | 8 | 4 | | | | 45 |
| % | 1.3 | 4.6 | 3.3 | 1.6 | 2.6 | 1.3 | | | | 14.7% |
| | | 8 | 2 | 1 | 2 | | | | | 13 |
| % | | 2.6 | 0.6 | 0.3 | 0.6 | | | | | 4.2% |

(²) E. GARDNER: *Phys. Rev.*, **75**, 379 (1949).

(⁴) D. SHERMANN: *UCRL*, 2726 (1954).

(³) C. J. WADDINGTON: *Phil. Mag.*, **45**, 1312 (1954).

(⁵) P. E. HODGSON: *Phil. Mag.*, **45**, 190 (1954).

in detail and the results were compared with the analogous results regarding the 94 MeV proton-stars.

The distributions of «projected» angles for doubly charged particles from α - and p-stars are shown in Fig. 1. No

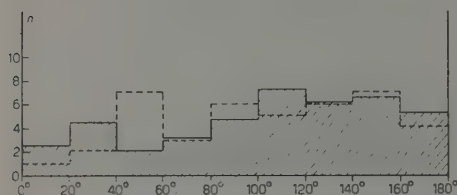


Fig. 1. — Angular distributions of doubly charged particles in the disintegrations of heavy elements produced by α 's of 350 MeV (broken line) and by protons of 94 MeV (continuous line).

significant difference is seen to exist between them. This seems to indicate that the process of nuclear excitation resulting in the acceleration of «evaporation» α -particles is not sensibly altered by this different mode of excitation.

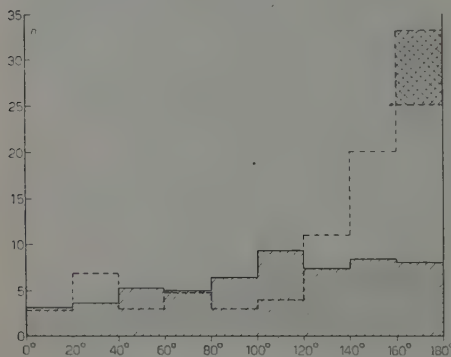


Fig. 2. — Angular distribution of singly charged particles in the disintegration of heavy elements produced by α 's of 350 MeV (broken line) and by protons of 94 MeV (continuous line).

In Fig. 2 the distributions of projected angles for singly charged particles are compared. Here the singly charged particles from α -particle disintegrations are much more grouped in the forward direction. Moreover the tracks having lowest grain densities appear at forward angles less than 20° . The double hatched portion of the last angular interval represents the particles, which have con-

served approximately the velocity of the incident α -particles.

Judging from this observation it would appear that some α -particle nucleons do not strongly interact, but traverse nuclear matter suffering only small angular deviations.

Out of a total of 306 nuclear disintegrations observed in this investigation 58 stars were observed in which one or two well-collimated singly charged particles came from the disintegration. Of these events 20 had secondary tracks sufficiently long so that accurate grain density and multiple scattering measurements could be made (length > 1 mm).

The results of these measurements are shown in Fig. 3, where each track is

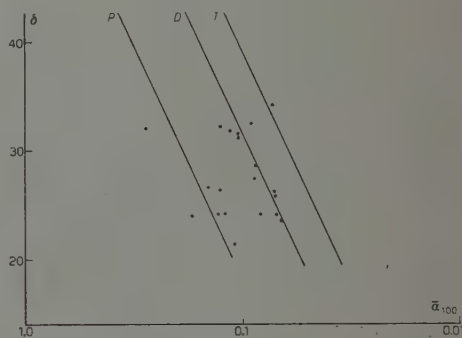


Fig. 3. — Relationship between grain density and scattering angle for singly charged particles coming out in the forward direction from disintegrations produced by α -particles.

represented by a point. For all tracks measured, the proton-deuteron-triton ratio is seen to be 7:12:1. Included in these measurements are three events each consisting of two singly charged secondaries of sufficient length to permit precise measurements and having no other charged secondary particles associated. The charged secondaries were identified in one case as two protons, in another as two deuterons and in the third as a proton and a triton. These events may be considered to be examples of the nuclear disintegration in flight of the α -particle, caused by its encounter with nuclear matter at the periphery of a target nucleus.

Collective Aspects of Nuclear Photoeffect.

U. L. BUSINARO

Laboratori CISE - Milano

S. GALLONE

Istituto di Fisica dell'Università e Laboratori CISE - Milano

(ricevuto il 16 Maggio 1955)

GOLDHABER and TELLER ⁽¹⁾ have proposed to interpret the resonance in nuclear photoeffect as due to excitation of a « dipole-type » oscillation in which the bulk of the protons is displaced with respect to the bulk of the neutrons. The restoring force acting between the proton and neutron spheres, was evaluated (see also ⁽²⁾), using a semiempirical argument, based on the symmetry term of binding energy.

Many works on nuclear photoeffect have been published afterwards. In ⁽³⁾ and other works, the suggestions contained in ⁽¹⁾ were developed according to hydrodynamical models. In the paper by REIFMAN ⁽⁴⁾, the resonance frequency is interpreted as due to single particle excitation and the resonance width as due to transfer of energy from single particle to collective modes of motion through surface interaction.

Here, an attempt is made to apply the collective model methods (see for instance ⁽⁵⁻¹⁰⁾), to nuclear photoeffect; unlike REIFMAN ⁽⁴⁾, we wish in analogy with the original idea of GOLDHABER and TELLER ⁽¹⁾, to consider the resonance as due to excitation of a collective mode and the width to its decay through single particle excitation.

In order to apply the collective model methods to the calculation of the restoring force acting in a motion similar to that considered in ⁽¹⁾, one may introduce the simplifying assumption that the depth of the average nuclear potential is proportional to the local value of the nucleonic density. Under this hypothesis, the perturbing potential associated with the displacement ξ of a sphere, — of radius equal

⁽¹⁾ M. GOLDHABER and E. TELLER: *Phys. Rev.*, **74**, 1047 (1948).

⁽²⁾ R. D. PRESENT: *Phys. Rev.*, **77**, 355 (1950).

⁽³⁾ H. STEINWEDEL and J. H. D. JENSEN: *Zeit. f. Nat.*, **5a**, 413 (1950).

⁽⁴⁾ A. REIFMAN: *Zeit. f. Nat.*, **8a**, 505 (1953).

⁽⁵⁾ T. S. LEVINGER: *Phys. Rev.*, **97**, 122 (1955).

⁽⁶⁾ J. RAINWATER: *Phys. Rev.*, **79**, 432 (1950).

⁽⁷⁾ A. BOHR and B. R. MOTTELSON: *K. Danske Vid.*, **27**, n. 14 (1953).

⁽⁸⁾ S. GALLONE and C. SALVETTI: *Nuovo Cimento*, **10**, 145 (1953).

⁽⁹⁾ D. L. HILL and J. A. WHEELER: *Phys. Rev.*, **89**, 1102 (1953).

⁽¹⁰⁾ G. SÜSsmANN: *Zeit. f. Nat.*, **138**, 543 (1954).

to nuclear radius R , containing φA uniformly distributed nucleons, with respect to a sphere of the same radius, containing the remaining $(1 - \varphi)A$ nucleons,—is given by:

$$(1) \quad H = -\varphi D \delta(r - R) \xi \mu - \varphi D \delta'(r - R) \frac{(\xi \mu)^2}{2},$$

where D is the depth of the unperturbed square well of radius R and μ is the cosine of the colatitude with respect to the direction of displacement.

The shift in the single particle level (n, l) due to the perturbation (1) is given by:

$$(2) \quad \Delta E_{nl} = \varphi(1 - \varphi) \frac{2E_{nl}}{3R^2} (1 + \kappa_{nl}R)\xi^2,$$

where E_{nl} and κ_{nl} may be approximated by the kinetic energy and wave number of the nucleon in the (n, l) shell, provided the potential well be sufficiently deep.

According to the collective model methods, the restoring force derives from the total energy variation obtained by summing (2) over all nucleons. One thus finds for the « dipole » frequency here considered:

$$(3) \quad \hbar\omega = \sqrt{\frac{4\hbar^2}{3M} \frac{\overline{E(1 + \kappa R)}}{R^2}}.$$

M is the nucleon mass and the bar indicates averaging over all nucleons. The frequency is independent from the factor $\varphi(1 - \varphi)$ because it also appears in the expression of the reduced mass.

Eq. (3) gives the correct order of magnitude for the resonance energy, and since the average values of kinetic energy and wave number are roughly independent from mass number A , one may argue from (3) that the resonance frequency varies according to a law which is in between $A^{-\frac{1}{2}}$ and $A^{-\frac{1}{3}}$. This last dependence, equal to that derived in ⁽¹⁾, predominates for not too light nuclei in accordance with experience ⁽¹¹⁾. However, if a harmonic oscillator potential is substituted to the square well potential used in the derivation of (3), one finds a collective frequency equal to that of the independent particle. One thus obtains ⁽⁵⁾ $\hbar\omega = 42 A^{-\frac{1}{2}}$ MeV, a dependence also obtained with the hydrodynamical model of STEINWEDEL ⁽³⁾.

If the assumption is made that the collective oscillation damps out mainly because of single particle excitation due to the interaction (1), one may attempt a standard calculation for the width. Assuming that the wave function of the system may be factorized ((see for analogy ⁽⁷⁾), into:

$$(4) \quad \psi(\xi) \Phi(\text{nucleons}),$$

where $\psi(\xi)$ is the collective oscillator wave function, the matrix element for a transition,—in which the collective oscillator falls from 1st to 0th oscillation, while a nucleon of shell (n, l) is risen to shell (n', l') , conserving its magnetic quantum

⁽¹¹⁾ R. MONTALBETTI, L. KATZ and J. GOLDBERG: *Phys. Rev.*, **91**, 659 (1953).

number m ,—factorizes into:

$$(5) \quad (1nlm|H|0n'l'm) = -\varphi D(1|\xi|0)(nlm|\delta(r-R)\mu|n'l'm),$$

with

$$(6) \quad (1|\xi|0) = \frac{\hbar}{\sqrt{2MA\varphi(1-\varphi)\hbar\omega}}$$

$$(7) \quad D(nlm|\delta(r-R)\mu|n'l'm) = \frac{2\sqrt{E_{nl}E_{n'l'}}}{R} (lm|\mu|l'm),$$

where E_{nl} is the same as in (2).

One has to calculate first the partial width Γ_{nl} , for excitation of a (n, l) shell nucleon:

$$(8) \quad \Gamma_{nl} = 2\pi \sum_{n'l'} \overline{(1nl|H|0n'l')^2} \delta(\hbar\omega - E_{n'l'} + E_{nl})$$

where the bar means average over m values. Using (5), (6) and (7), one gets:

$$(9) \quad \overline{(1nl|H|0n'l')^2} = \frac{\hbar^2}{M\hbar\omega} \frac{\varphi}{1-\varphi} \frac{E_{nl}E_{n'l'}}{R^2} \frac{1}{3} \frac{l+1}{2l+1} \quad (l' = l \pm 1).$$

The density of final states may be calculated from the asymptotic expressions of energy levels for an infinitely deep square well potential:

$$(10) \quad E_{n'l'} \cong \frac{\pi^2 \hbar^2 (n' + l'/2)^2}{2MR^2}$$

which yields

$$(11) \quad \left(\frac{\partial E_{n'l'}}{\partial n'} \right)^{-1} = \sqrt{\frac{MR^2}{2\hbar^2 \pi^2}} \cdot \frac{1}{\sqrt{E_{n'l'}}}.$$

One then gets for Γ_{nl} :

$$(12) \quad \Gamma_{nl} = \frac{\varphi}{1-\varphi} \hbar \sqrt{\frac{2}{M}} \frac{E_{nl} \sqrt{E_{nl} + \hbar\omega}}{AR\hbar\omega} \cdot \frac{4}{3} \frac{l+1}{2l+1}.$$

Taking roughly $\frac{4}{3} \frac{l+1}{2l+1} \cong 1$ and summing over all nucleons, one has for the total width:

$$(13) \quad \Gamma \cong \frac{\varphi}{1-\varphi} \hbar \sqrt{\frac{2}{M}} \frac{E \sqrt{E + \hbar\omega}}{R\hbar\omega},$$

where the bar indicates averaging as in (3). Γ results roughly independent from A , and putting $\varphi = \frac{1}{2}$, $E = \hbar\omega = 10$ MeV, $A^{\frac{1}{3}} = 6$, one gets the order of magnitude $\Gamma \sim 4$ MeV, in agreement with experiment ⁽¹¹⁾.

A calculation following the same lines and leading essentially to the same results may be performed using an oscillator potential instead of a square well potential.

One may generalize somewhat the above calculations by considering types of motions in which φA nucleons, uniformly distributed in the varying boundary

$$(14) \quad r(\mu) = R \left[1 + \alpha_j + \sum_{i=1}^{\infty} \alpha_i P_i(\mu) \right],$$

oscillate with respect to $(1 - \varphi)A$ nucleons enclosed in a sphere. The volumes of this sphere and that of the displaced matter remain both constant and equal to the initial nuclear volume.

Up to first order terms in φ , the energy shift for a (n, l) nucleon is given by:

$$(15) \quad \Delta E_{nl} = 2\varphi E_{nl} \left[- \sum_i \alpha_i \langle P_i \rangle + \sum_i \frac{\alpha_i^2}{2i+1} + \kappa_{nl} R \sum_{i,s} \alpha_i \alpha_s \langle P_i P_s \rangle \right]$$

where $\langle P_i P_s \rangle = \int Y_{lm}^* P_i P_s Y_{lm} d\Omega$, and E_{nl} and κ_{nl} are as in (3).

Neglecting the contribution of incomplete shells, one obtains for the total potential energy:

$$(16) \quad \Delta E = 2\varphi \overline{E(1 + \kappa R)A} \sum_i \frac{\alpha_i^2}{2i+1},$$

the bar indicating average value on all nucleons.

The kinetic energy T may for instance be calculated by treating the displaced nucleons like an incompressible fluid in irrotational motion. One has:

$$(17) \quad T = \varphi A M \frac{3R^2}{2} \sum_{i=1}^{\infty} \frac{\alpha_i^2}{i(i+1)}.$$

The frequency for the i -th collective mode is thus given by:

$$(18) \quad \hbar\omega = \sqrt{\frac{4\hbar^2}{3M} i - \frac{E(1 + \kappa R)}{R^2}}.$$

Provided this result be still valid for a large φ one may ask why the frequencies given by (18) are so much higher than those calculated in the usual hydrodynamical incompressible model. Frequencies comparable with those associated with surface waves in hydrodynamical models may be explained⁽⁸⁻¹⁰⁾ by re-arrangements of the single particle states which maintain minimum deformation energy. This argument is of course valid only if the period of collective motion is large as compared with that of the single particle. This is certainly not the case in the range of photo-effect resonance, where both periods are comparable. Here, «slippage» does not take place and the potential to be considered is the one associated with a rapid collective deformation⁽⁹⁾, giving rise to high frequencies of the type calculated above.

Quantum Field Theory and the Two-Body Problem of General Relativity.

E. CORINALDESI

Dublin Institute for Advanced Studies

(ricevuto l'11 Maggio 1955)

Recently GUPTA ⁽¹⁾ put forward the view that Einstein's theory of gravitation can be reinterpreted as a theory in flat space with a Lagrangian density consisting of an infinite number of terms. On the basis of this idea, he has evolved a practicable method of quantization of the gravitational field in interaction with the electromagnetic field, which is an extension of Rosenfeld's early work on the quantization of the linearized Einstein equations.

In this note we shall endeavour to present, as a proof of the soundness of Gupta's approach, a field-theoretical derivation of the equations of the two-body problem of general relativity, which were found by EINSTEIN, INFELD and HOFFMANN ⁽²⁾ (the HEI equations).

Consider the Lagrangian density

$$L = -\frac{1}{4} \left(\gamma_{\mu\nu,\lambda} \gamma_{\mu\nu,\lambda} - \frac{1}{2} \gamma_{,\lambda} \gamma_{,\lambda} \right) + \sum_{i=1}^2 \left(L^{(i)} - \frac{1}{2} \kappa h_{\mu\nu} T_{\mu\nu}^{(i)} \right)$$

with $h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2} \gamma \delta_{\mu\nu}$. Here $\gamma_{\mu\nu}$ and $\gamma = \gamma'_{\mu\mu}$ represent the gravitational field, and $L^{(i)}$, $T_{\mu\nu}^{(i)}$ are the Lagrangians and the energy-momentum tensors of two fields of scalar particles of mass m_1 and m_2 . Terms of higher order in κ have been neglected as they are not necessary for our purpose. The Lagrangian in its entirety would be equivalent to the general theory of the gravitational field co-existing with the two scalar fields.

From the Lagrangian it is easy to deduce the interaction Hamiltonian in the interaction representation, and to calculate, by the Feynman-Dyson method, the matrix element for scattering of a particle m_1 by a particle m_2 due to the exchange of one graviton. The matrix element thus obtained, $(\mathbf{p}_1 \mathbf{p}_2 | V | \mathbf{p}'_1 \mathbf{p}'_2)$, is translated into configuration space by a Fourier transformation, and defines a velocity dependent potential $(\mathbf{r}_1 \mathbf{r}_2 | V | \mathbf{r}'_1 \mathbf{r}'_2)$, of the order of κ^2 , which reduces to the Newton

⁽¹⁾ S. N. GUPTA: *Proc. Phys. Soc. (London)*, A **65**, 161 608 (1952); *Phys. Rev.*, **96**, 1683 (1954).

⁽²⁾ A. EINSTEIN, L. INFELD and B. HOFFMANN: *Ann. Math.*, **39**, 66, (1938); Cf. also B. BERTOTTI: *Nuovo Cimento*, **12**, 226 (1954).

potential if the velocity dependent terms are neglected. This potential is then used in the two particle Hamiltonian

$$H = \frac{1}{2m_1} \mathbf{p}_1^2 + \frac{1}{2m_2} \mathbf{p}_2^2 - \frac{1}{8m_1^3} (\mathbf{p}_1^2)^2 - \frac{1}{8m_2^3} (\mathbf{p}_2^2)^2 + \dots + V$$

by the help of which the second time derivative of the coordinate r_1^i is evaluated by the formula

$$\ddot{r}_1^i = -[H, [H, r_1^i]].$$

The calculation has actually been performed in the coordinate representation, and the matrix element $\langle \mathbf{r}_1 \mathbf{r}_2 | \ddot{r}_1^i | \mathbf{r}'_1 \mathbf{r}'_2 \rangle$ has been found.

Take now a wave function $\Psi(\mathbf{r}_1 \mathbf{r}_2)$ representing two spatially separated wave packets for the two particles m_1 and m_2 . The expectation value

$$\langle \ddot{r}_1^i \rangle = \int \Psi^*(\mathbf{r}_1 \mathbf{r}_2) \langle \mathbf{r}_1 \mathbf{r}_2 | \ddot{r}_1^i | \mathbf{r}'_1 \mathbf{r}'_2 \rangle \Psi(\mathbf{r}'_1 \mathbf{r}'_2) d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}'_1 d\mathbf{r}'_2$$

is, by a series of partial integrations and by neglecting terms more than quadratic in the velocities, found to reduce to

$$\begin{aligned} \langle \ddot{r}_1^i \rangle = & \kappa^2 \int \Psi^*(\mathbf{r}_1 \mathbf{r}_2) \left\{ \varphi_{,i} + v_1^s \varphi_{,i} v_1^s - 2[v_1^s \varphi_{,s} v_1^i + v_1^i \varphi_{,s} v_1^s] - \right. \\ & - 2[v_1^s \varphi_{,i} v_2^s + v_2^s \varphi_{,i} v_1^s] + 2[v_1^s \varphi_{,s} v_2^i + v_2^i \varphi_{,s} v_1^s] + \\ & + \frac{3}{2} [v_1^i \varphi_{,s} v_2^s + v_2^s \varphi_{,s} v_1^i] + 2v_2^s \varphi_{,i} v_2^s - 2[v_2^i \varphi_{,s} v_2^s + v_2^s \varphi_{,s} v_2^i] - \\ & \left. - \frac{m_2}{2} v_2^s \left(\frac{\partial}{\partial r_1^i} \left(\frac{(r_1^s - r_2^s)(r_1^n - r_2^n)}{r^3} \right) \right) v_2^n \right\} \Psi(\mathbf{r}_1 \mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2. \end{aligned}$$

Here $r = |\mathbf{r}_1 - \mathbf{r}_2|$, $\varphi = \frac{m_2}{r}$, $\varphi_{,i} \equiv \frac{\partial}{\partial r_1^i} \varphi$ and $v_1^s \equiv \frac{1}{im_1} \frac{\partial}{\partial r_1^s}$ etc..

But for two velocity independent terms, this formula coincides with the HEI equations, if these latter are written in a symmetrized form so as to satisfy the requirement of hermiticity. The two missing terms are of a higher order than κ^2 . They can be obtained from a Lagrangian including terms in κ^2 and the next Feynman diagrams, and will be included in a detailed paper shortly to be published.

LIBRI RICEVUTI E RECENSIONI

A. DAUVILLIER - *Les Rayons Cosmiques*; due volumi di XII+248 e XII+318 p., Ed. Dunod, Paris 1954; 4350 Fr.

La compilazione di un trattato generale sui Raggi Cosmici, che vuole essere, per di più, una rassegna quasi completa di tutta l'enorme mole di ricerche sperimentali e teoriche degli ultimi cinquanta anni, è certamente una impresa da far tremare «le vene e i polsi» a chiunque, anche provvisto di una esperienza specifica pluridecennale e di una vasta cultura scientifica. L'una o l'altra (o più insieme) di alcune limitazioni a priori sembrano praticamente inevitabili per chi si accinge all'impresa: servirsi della collaborazione di molte persone, specialiste ciascuna in un più o meno ristretto settore teorico o sperimentale; limitare l'esposizione alle conclusioni più recenti in ciascun settore, rinunciando alla discussione storico-cronologica; restringere l'obbiettivo ad un campo nettamente circoscritto. Ad esse si deve in ogni caso accompagnare, affinché l'opera risulti di agevole consultazione, un poderoso sforzo di organizzazione ed ordinamento della materia; sforzo che si deve manifestare anche esteriormente in una minuta suddivisione in capitoli, paragrafi e sottoparagrafi, corredata da indici esaurienti (in particolare un completo indice analitico ed una bibliografia bene ordinata). Dalla applicazione di questi criteri ci sembrano essere esempi notevoli i recenti, ma già classici testi di Heisenberg e di Rossi e i volumi *Progress in Cosmic Ray Physics* della North Holland Publ. Company.

L'aver voluto affrancarsi (ad ogni

costo, si direbbe) da queste limitazioni costituisce, a nostro avviso, la singolarità ed al tempo stesso il difetto di origine del recente trattato di A. Dauvillier. È ben vero che il sottotitolo dell'opera dichiara che i Raggi Cosmici sono qui studiati «dans leurs rapports avec l'Electricité Atmosphérique, la Météorologie, le Géomagnétisme et l'Astronomie»; ma il contenuto dei due volumi sconfina a più riprese fuori di questi limiti, di per sé già tanto vasti anche perché proprio in quei campi si è a tutt'oggi ben lontani da risultati sia pur approssimativamente conclusivi. Si veda ad esempio il lungo capitolo (il secondo) sulle tecniche di misura, in alcune parti incredibilmente minuzioso e comunque troppo vasto per chi desidera un panorama rapido dei metodi sperimentali, ma contemporaneamente (e necessariamente) incompleto e non aggiornato per chi desideri una messa a punto sugli sviluppi tecnici più recenti.

La mole del materiale raccolto è veramente impressionante; la bibliografia conta circa 1500 richiami (ma purtroppo manca un indice bibliografico) e quasi tutti i lavori sono non solamente citati, ma riassunti e commentati nel testo, anche quando si tratta di esperimenti e di ipotesi da tempo superati, che hanno ormai soltanto un interesse storico. Le informazioni, almeno per quel che riguarda gli aspetti sperimentali dei problemi trattati, appaiono complete ed aggiornate, tranne che per il già citato capitolo sulle tecniche. Ma il metodo di esposizione è quasi sempre quello storico-cronologico, mentre ci sembra mancare quasi del tutto quello sforzo di organizzazione logica e pratica di cui abbiamo

parlato in principio; e questo rende difficile la consultazione dell'opera, ed ancor più il suo uso come testo di studio e di aggiornamento sui problemi connessi con la Radiazione Cosmica.

Un'opera di questo genere è anche inevitabilmente soggetta a contenere un certo numero di inesattezze. Spigolando a caso, segnaliamo a pag. 369, la singolare affermazione che il mesone μ non sia capace di irraggiare luce di Čerenkov; alla pagina seguente, la discussione (che ci sembra alquanto oscura) delle proprietà del « neutretto » e del suo ruolo nella famiglia delle particelle elementari; a pag. 416 una ovvia confusione fra effetto Čerenkov e fluorescenza.

Nonostante questi difetti, il libro di Danvillier presenta un notevole interesse sotto almeno due punti di vista: in primo luogo l'ampiezza ed il rilievo dati alle correlazioni fra la fisica dei Raggi Cosmici e la Geofisica, la Meteorologia, l'Elettricità atmosferica; il fisico non specializzato in questi campi trova qui raccolta per la prima volta una messe di informazioni interessanti, che sarebbe assai faticoso procurarsi direttamente. In secondo luogo, proprio il metodo storico di esposizione, e la insistenza (dianzi criticata come eccessiva) su ipotesi, polemiche, descrizioni di tecniche e strumenti superati da tempo, danno all'opera un certo singolare fascino, che probabilmente rispecchia la personalità dell'Autore, e la rendono una preziosa fonte per chi voglia formarsi un'idea del faticoso e spesso erratico progredire della conoscenza in questo campo della

Fisica, o addirittura per un eventuale studioso della storia delle tecniche sperimentali della Fisica Nucleare.

La materia è suddivisa in otto capitoli: il primo è dedicato ad una esposizione delle relazioni fra i Raggi Cosmici e la elettricità atmosferica, con particolare riguardo al problema della carica elettrica della Terra; il secondo tratta delle tecniche sperimentali (camere di ionizzazione, contatori di Geiger, contatori a scintilla, contatori a fluorescenza, circuiti elettronici di selezione e conteggio, camere a nebbia ad espansione e a diffusione, emulsioni fotografiche nucleari); il terzo espone i principali risultati sperimentali sull'assorbimento dei Raggi Cosmici nell'atmosfera e nei materiali condensati. Il quarto capitolo è dedicato agli effetti geomagnetici, mentre nel quinto sono raccolte, un po' disordinatamente, informazioni sulla natura dei Raggi Cosmici, con qualche cenno alle teorie dei processi moltiplicativi, sia elettromagnetici che nucleari. Il sesto capitolo tratta del bilancio energetico della radiazione cosmica nell'atmosfera e degli effetti (radioattività indotte, effetti biologici) che sono o si pensa potrebbero essere da essa provocati; nel settimo sono discusse le variazioni nel tempo della intensità delle varie componenti, con particolare riguardo alle correlazioni meteorologiche. L'ottavo capitolo, infine, è dedicato ai problemi dell'origine dei Raggi Cosmici e del loro ruolo nella evoluzione dell'Universo.

L. MEZZETTI